THE EFFECTS OF AN ADVANCED HIGH SCHOOL COURSE IN NUMBER THEORY AND ALGEBRA ON STUDENTS' MATHEMATICAL SELF-EFFICACY

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ABSTRACT

The purpose of this mixed methods study was to determine the effects of participation in an advanced course in proofs and problems in number theory and algebra on high achieving high school students' mathematical self-efficacy. In addition to learning how to prove theorems, students were expected to engage in higher order mathematical thinking for the purpose of developing mathematical habits of mind. Because the course focused on abstract thinking, and thus was significantly different from the other math courses students had taken, the study focused on determining whether self-efficacy changed as a result of course participation, particularly because of the ways self-efficacy affects goal setting and perseverance in the face of challenging tasks. To that end, self-efficacy was measured at the beginning and end of the course using a self-efficacy instrument aligned with course goals. In addition, students participated in group interviews at the end of the course and provided written feedback about ways course participation affected their self-efficacy as well as their interest in pursuing additional advanced math courses in college. Results indicated a large effect size difference between students' pre- and post-course course self-efficacy as well as their self-efficacy at the beginning of the course and their perceived ability to complete course goals (potential). Further, although all participating students earned an A in the course and demonstrated their attainment of the course objectives, qualitative data revealed that students characterized themselves as either in the group who "got" PPNTA or who "didn't get" the course [Note: "get it" is an American idiom that means to deeply understand

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something]. Those who placed themselves in the group who "got it" had higher self-efficacy, were more interested in the abstract, theoretical aspects of the course, and demonstrated greater interested in taking advanced math courses in college. Students in the group that "didn't get" the course described fluctuations in their self-efficacy that were dependent on the difficulty of the topics being covered. In addition, these students were more interested in taking courses that focused on concrete knowledge and practical application, and they were less interested in pursuing advanced math courses in college.

Keywords: number theory, mathematical proofs, self-efficacy.

INTRODUCTION

In this study, we investigated changes in high school students' mathematical self-efficacy as they participated in an advanced course titled *Proofs and Problems in Number Theory and Algebra* (PPNTA). The main goals of the course were for students to employ higher order mathematical thinking skills, develop ability in proof construction and problem solving, recognize when a proof is incorrect, and gain an understanding of the concept of a mathematical habit of the mind. Students were also expected to engage in intellectual argument about mathematics with their peers and to work individually and in collaborative teams to solve complex problems. Course goals, which focused on abstract ideas such as elegance in proofs and solutions as well as metamathematics, were significantly different from the content-specific goals of the other advanced math courses students had completed during high school.

The mathematically gifted students in this study had demonstrated extraordinary achievement throughout their school careers, and they self-reported high levels of self-efficacy in previous advanced math classes that focused on content-specific objectives. However, the kind of abstract thinking necessary for writing proofs is significantly different than the conceptual understanding and manipulation required to study a topic such as calculus. Thus, in this study we wished to determine whether self-efficacy beliefs would remain high in the PPNTA course, which had much different course goals than students' previous math courses and emphasized constructing proofs and engaging in mathematical habits of the mind (MHM).

Though an agreed upon definition of MHM does not yet exist in the literature, Millman and Jacobbe (2008, 2009) suggest that it includes (1) exploring mathematical ideas, (2) formulating questions, (3) constructing examples, (4) identifying problem solving approaches that are useful for large classes of problems, (5) asking whether there is "something more" (a generalization) in the mathematics on which students are working, and (6) reflecting on answers to see whether an error has been made. Cuoco, Goldenberg, and Mark (1996), in describing the habits of mind of mathematicians, explain that MHM includes:

1. learning to recognize when problems or statements that purport to be mathematical are, in truth, still quite ill-posed or fuzzy,
2. becoming comfortable with and skilled at bringing mathematical meaning to problems and statements through definition, systematization, abstraction, or logical connection making, and
3. seeking and developing new ways of describing situations. (p. 376).

The habits of mind proposed by Millman and Jacobbe and Cuoco et al. are in line with the National Council of Teachers of Mathematics (NCTM) Standards (2000), which emphasize the need for teachers to help students become mathematic problem solvers and develop "reflective habits of mind by asking questions such as, 'Before we go on, are we sure we understand this?' 'What are our options?' 'Do we have a plan?' 'Are we making progress or should we reconsider what we are doing?' 'Why do we think this is true?'" (p. 54). The MHM elements described here also are closely linked to Pólya's principles of problem solving (Pólya, 1945).

Because self-efficacy plays a critical role in the ways individuals approach difficult tasks, set goals for themselves, and persevere when faced with a challenging problem (Bandura, 1994)—and because these self-regulation strategies were so closely aligned with the elements of mathematical habits of the mind—we examined the self-efficacy beliefs of students at the beginning of the PPNTA course and near the end of the course to determine whether students' self-efficacy was affected by course participation. We also interviewed students about their experiences in the course to examine more deeply changes in self-efficacy beliefs and what students perceived to be the reasons for those changes.

THEORETICAL FRAMEWORK

Self-efficacy refers to the self-referent beliefs individuals hold about their capability to achieve a certain level of performance on a given task or goal (Bandura, 1994). These beliefs affect individuals in a number of ways. As Bandura (1997) explains,

> *Such beliefs influence the courses of action people choose to pursue, how much effort they put forth in given endeavors, how long they will*

> *persevere in the face of obstacles and failures, their resilience to adversity, [and] whether their thought patterns are self-hindering or self-aiding... (p. 3)*

Thus, in academic settings, students with high self-efficacy will engage in the types of behaviors that lead to achievement, whereas individuals with low self-efficacy will engage in behaviors that can undermine success (Pajares, 1996). Because self-efficacy influences the actions one takes, it is often a better predictor of what a person can accomplish than are his or her actual capabilities (Pajares & Schunk, 2001).

Research has consistently demonstrated a positive relationship between self-efficacy and academic achievement. For example, in their meta-analysis of 38 self-efficacy studies at the elementary, high school, and college levels conducted between 1977 and 1989, Multon, Brown, and Lent (1991) found a positive correlation between self-efficacy beliefs and academic performance, with self-efficacy accounting, on average, for about 14% of the variance in students' academic achievement. Effects were slightly larger for high school students, with self-efficacy accounting for 17% of the variance in student achievement.

A number of researchers have also conducted studies specifically on mathematics self-efficacy, and results consistently show a relationship between self-efficacy and mathematics achievement. In a study of high school students conducted by Pajares and Kranzler (1995), the researchers found math self-efficacy to have a strong, direct effect on mathematics problem-solving, even when the researchers controlled for general mental ability. In their study of middle school students, Pajares and Graham (1999) found similar results, concluding that math self-efficacy made an independent contribution to math performance when variables such as anxiety, self-concept, self-regulation, engagement, and prior academic achievement were controlled. The relationship between self-efficacy and achievement has also been demonstrated with college students. Pajares and Miller (1994), for example, measured undergraduates' math self-efficacy, math self-concept, math anxiety, perceived usefulness of math, prior math experience, and math performance. Results indicated that math self-efficacy was a better predictor of math performance than were the other variables measures.

The association between self-efficacy and achievement is seen as a reciprocal relationship. As Pajares and Schunk (2001) explain, "According to Bandura's social cognitive theory, behavioral and environmental information create the self-beliefs that, in turn, inform and alter subsequent behavior and environments" (p. 251). These self-beliefs differ between gifted and non-gifted students. Studies by Pajares (1996), Pajares and Graham (1999), Pajares and Kranzler (1995), Zimmerman, Bandura, and Martinez-Pons (1992) have found that higher ability students have a stronger sense of self-efficacy and their self-perceptions are more accurate than are those of average ability students. Non-gifted students tend to be overconfident in their ability (Pajares, 1999). Another difference between gifted and non-gifted students relates to influences on self-efficacy. As revealed in Pajares' (1996) study, gifted middle school students' math self-efficacy was directly influenced by their cognitive ability but was not influenced by their prior achievement. The opposite was found for regular education students, whose self-efficacy was directly affected by achievement but not by cognitive ability. Pajares (1996) suggests that because gifted students rely more on ability than on performance to gauge their self-efficacy, their beliefs "may be more stable and resilient, being based on the positive self-perceptions of ability borne of their identification as gifted" (p. 339).

METHODS

PPNTA Course

We developed this course in partnership with a local charter high school for math, science and technology. In this collaborative effort, Georgia Institute of Technology (Georgia Tech) worked with the school to design the course, and Georgia Tech's Center for Education Integrating Science, Math, and Computing (CEISMC) financially supported course instruction through a research assistantship funded through Georgia's Race to the Top award from the U.S. Department of Education. A Georgia Tech graduate student in mathematics and computer science, who had completed his bachelor's degree in mathematics with highest honors, was the instructor for the course. Prior to teaching the PPNTA course, he served two years as a teaching assistant for undergraduate calculus courses. The director of CEISMC (a co-author of this paper) had a major role in working with the course instructor to plan the PPNTA course. In addition, he made several classroom observations throughout the semester and delivered class lectures on introduction to proofs, the notion and use of equivalence classes, and introduction to the topology of matrix groups. He was also the co-author of the manuscript used in the course.

The 18-week course was designed to introduce students to mathematical proofs using number theory and algebra as the contexts of study. Guiding course development was our desire for students to understand that mathematics is not based on rote memorization of facts nor is it fundamentally about calculation. Thus, we introduced students to mathematics as a living research discipline they could use for discovering new ideas about numbers, space, and functions as well as the interrelationships among these ideas. In the course, we wanted to create an *explore, generalize, prove, think* environment that required students to approach the study of mathematics in fundamentally different ways than they had in traditional math courses. We conceived this environment as similar to the culture of that of mathematics researcher in general, recognizing the difference between a group of mature mathematicians and a group of gifted, high achieving math students.

To illustrate the way course activities were structured to encourage students' exploration of math, we provide an example of a class activity covered the third week of class. First, the teacher worked with students to demonstrate that the $\sqrt{2}$ is irrational using the usual proof by contradiction. Following this, students were asked to construct a proof of the fact that $\sqrt{3}$ is irrational using the logic of the $\sqrt{2}$ example. From this approach, they were asked to generalize the procedure so that it was valid for the square root of any prime number. In order to ensure students actually understood what was going on in this proof structure, we next asked them to prove that the $\sqrt{4}$ was irrational. Although students knew this was false, having the students figure out why the "proof" of $\sqrt{4}$ is irrational must be incorrect was important for them to truly understand what a proof is—and what it isn't. This approach to proof demonstrates use of mathematical habits of mind.

The course covered a number of topics including basic properties of integers; divisibility and prime numbers; the Fundamental Theorem of Arithmetic; linear Diophantine equations; equivalence relations and their applications; basic properties of polynomials; divisibility of polynomials, divisibility methods, and roots of polynomials; combinatorics; elementary group theory; public key cryptography; and problem solving with computer programming. Throughout the study of these topics, broad goals for students were to (a) identify what makes a mathematical proof correct, (b) learn commonly applied proof techniques, (c) develop proficiency in reading and writing mathematics in general and proofs in particular, and (d) apply problem-solving methods to find solutions and demonstrate the correctness of their methods.

In addition to these goals, we developed these specific course objectives for students:

1. Understand the importance of proofs in mathematics;
2. Learn different methods to construct proofs;
3. Understand the concept of "elegance" in proofs;
4. Create examples that provide insight into designing proofs;
5. Construct valid proofs;
6. Identify the fallacious reasoning in incorrect proofs;
7. Engage in intellectual arguments with others about mathematics;
8. Explain ideas that motivate proofs;
9. Use concepts learned about elementary number theory and algebra in other courses to solve problems;
10. Work individually to solve mathematical problems from number theory and algebra;
11. Work in teams to solve mathematical problems from number theory and algebra; and
12. Define what "mathematical habit of mind" means.

The PPNTA course was completed during an 18-week semester that started in January and concluded in May. Each class period was 48 minutes. Class activities included instructor lecture, instructor review of problem sets, independent and group problem-solving, and student-led presentations to the class.

Participants

Nineteen students were enrolled in the course, and 15 provided parental consent and/or assent to be in the study. Fourteen of the participants were seniors in their final semester of high school, and 1 student was a junior. Eleven were male, and 4 were female. The majority of students (10) were of Asian or East Asian descent, and 5 were Caucasian. All 15 students earned an A as a final grade in the PPNTA course. Participants had completed every available high school math course except AP Statistics, and all but one had completed Calculus II and Calculus III at Georgia Tech (courses that follow AP Calculus). Participating students' average high school GPA was 3.95, and their overall high school math GPA (including the Georgia Tech calculus courses) was 3.99. There was only one student in the group whose overall math GPA was not 4.0. Six students had perfect scores (800) on the math section of the Scholastic Aptitude Test (SAT). Participating students' average math SAT score was 781; the average SAT score, which includes math, critical reading, and writing sections, was 2221, putting them at the 99th percentile for all college-bound SAT test takers. Students' average SAT scores were at the 99th percentile in math and the 97th percentile in both critical reading and writing. All 15 students were planning on majoring in a premedical or STEM field in college.

Setting

The study was conducted in a public, charter high school for mathematics, science, and technology that was established in 2007. At the time of the study, the school enrolled just over 500 students in grades 9 through 12, including 38% Asian, 32% Caucasian, 16% African-American, and 10% Hispanic students. Approximately 76% of students were in gifted programs. For the 2009-2010 school year, 100% of 11th grade students met or exceeded statewide achievement standards in all academic subjects.

The new campus, which was opened in 2010, is designed for project-based collaborative work among students, universities, and the business community. Any eighth grade student in the county may apply for admission. Because of the high number of applicants each year, students are admitted based on lottery selection. Once students are admitted to the school, they choose one of three areas of study—

engineering, biosciences, or emerging technologies. The school offers 15 Advanced Placement (AP) courses, including those in calculus, statistics, physics, biology, chemistry, computer science, and humanities. Also offered are courses in accelerated integrated geometry and accelerated integrated pre-calculus, Calculus II and III (taught through video conferencing as dual-enrollment courses with Georgia Tech), differential equations, and PPNTA.

Data Collection

To determine how participation in the PPNTA course affected students' mathematical self-efficacy, we measured self-efficacy using a pre and post math self-efficacy scale we developed for this study. As we developed the scale, we relied on Bandura's (2006) *Guide for Constructing Self-Efficacy Scales* and created items that were tied to specific course objectives (see Appendix 1). Based on Bandura's guide, we phrased items on the pre- and posttest in terms of what students *can* do in order to measure perceived capability. We also asked students to rank how confident they were that they would be able to achieve each goal; thus, on the pretest, students ranked their current confidence for each course goal as well as how confident they were that they could reach each course goal.

Although Bandura (2006) emphasized the importance of creating self-efficacy instruments that measure an individual's current operative capabilities rather than their potential, we extended that measure so that students ranked both current operative capability and potential capability. Because students had not yet had the opportunity to develop ability in many of the course goals, we believed self-efficacy would be low on the pretest and likely would be higher on the posttest. Thus, measuring self-efficacy in terms of both current and potential capability, and then comparing those measures to posttest scores, would allow us to make additional comparisons that would increase the meaningfulness of results, particularly with our gifted population.

On the pretest, students ranked their self-efficacy, on a scale of 0 to 100 (with 100 indicating high efficacy), on 12 items tied to course goals. We chose this scale based on Bandura's suggestion to use a broad range to increase measurement sensitivity and reliability. Students ranked how confident they were that they could already complete each task or goal (e.g., *understand the importance of proofs in mathematics*) and how confident they were that they could reach each goal. The posttest included identical self-efficacy items as did the pretest; however, students only ranked their confidence in their current ability to complete each task.

At the end of the course, we also asked students to complete an 11-item post-course survey about their experiences in the course. On the survey, students provided demographic information and perceptions of the course, and they were asked to describe ways, if any, their self-efficacy had changed during the course and the aspects of the course that affected their self-efficacy. All students in the study provided open-ended responses to these questions.

Finally, in the last week of school, the researcher who had not engaged in any instruction during the course (and with whom students had not had contact during the course) conducted focus group interviews with all but one of the participating students (who was not available due to scheduling conflicts). Students were interviewed in groups of 2 to 6 students, except for one interview that took place with an individual who wished to be interviewed separately. Interviews lasted between 40 an 60 minutes. In the semi-structured interviews, students were asked to describe their experiences in the course as well as their perceptions of the course. In those cases where issues of self-efficacy were not described by students, the researcher asked if students' confidence had been affected in the course, and if so, to describe how. All interviews and focus groups were transcribed, and then data were analyzed following Yin's (2011) five-phased cycle of compiling, disassembling, reassembling, interpreting, and concluding. After data were transcribed and compiled, they were coded based on patterns observed in the data. Following this disassembling of data, we reassembled the data into themes based on the observed patterns. Interpretations and conclusions are provided in the following section.

RESULTS AND DISCUSSION

Because this study was classroom-based research with a small sample (n=15), we present mean differences between pretests and posttests and provide standardized effect sizes using Cohen's *d*, which reveal the magnitude of the differences in standard deviation units. First, we compared overall self-efficacy means for (1) the pretest score of students' current ability to complete each course goal (prior to beginning the course), (2) the pretest score of students' confidence they could reach each course goal (potential capability), and (3) the posttest score of students' confidence in their ability in each course goal area measured at the end of the course.

As shown in Table 1, there was a 32.5-point difference between the pre-measures, indicating that although students began the course with low self-efficacy ($\bar{x}$ = 47.02), their efficacy regarding their ability to learn content and accomplish the course goals was significantly higher ($\bar{x}$= 79.52). The standardized effect size for this comparison was 2.15, revealing a difference of over two standard deviations between students' current ability and potential capability at the beginning of the course.

Table 1. Pretest, Potential, and Posttest Comparisons of Students' Self-Efficacy

	N	$\bar{x}$	SD	Diff.	Cohen's *d*
Pretest (Current Ability)	15	47.02	17.16	32.50	2.15
Pretest (Potential Capability)	15	79.52	13.14		
Pretest (Potential Capability)	15	79.52	13.14	-3.34	-0.23
Posttest (Current Ability)	15	76.18	16.06		
Pretest (Current Ability)	15	47.02	17.16	29.16	1.76
Posttest (Current Ability)	15	76.18	16.06		

There was a 3.34-point difference between students' potential capability at the beginning of the course and self-efficacy in their ability at the end of the course, and the effect size for this difference was small (-0.23). Comparisons of pre- and posttest current abilities revealed a large effect size (1.76). Students had much higher self-efficacy at the end of the course than they did at the beginning. Pre-post differences in students' potential and current ability are graphically displayed in Figure 1.

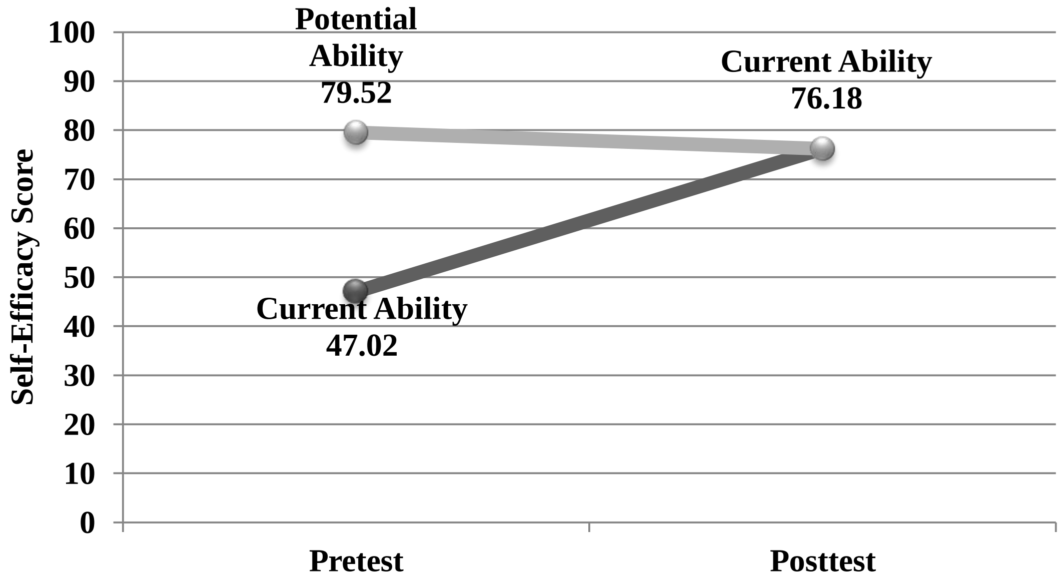

Figure 1. Pre and Post Self-Efficacy Comparisons

When comparing gender and racial differences across self-efficacy measures, there were differences between the groups. As illustrated in Figure 2, males and females had nearly identical average scores on the self-efficacy pretest of current ability. However, males had higher scores on the self-efficacy pretest of potential ability and on the posttest.

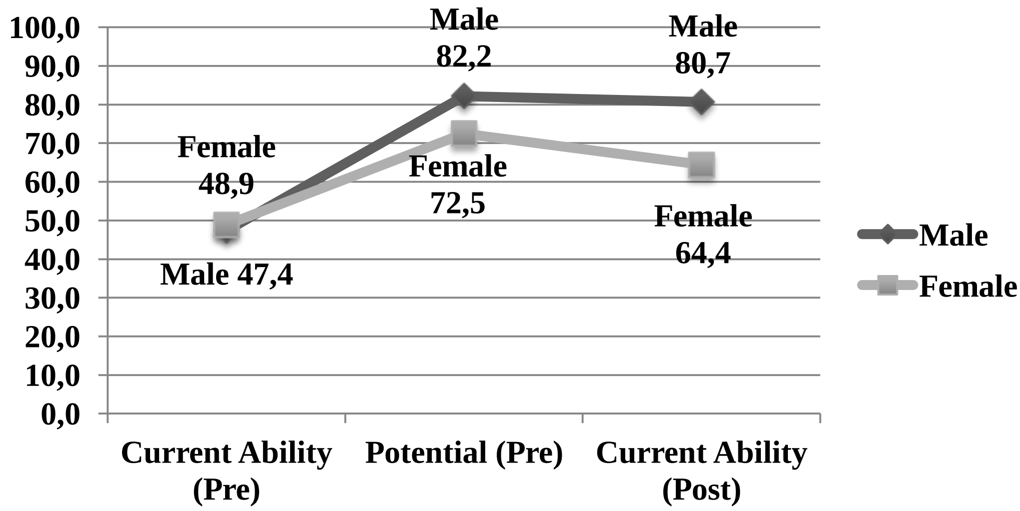

Figure 2. Pre and Post Self-Efficacy Comparisons by Gender

Figure 3 displays self-efficacy score differences between Asian and Caucasian students. Caucasian students had higher self-efficacy on the pretest of current ability and on the posttest. Self-efficacy was nearly identical on the pretest of potential ability.

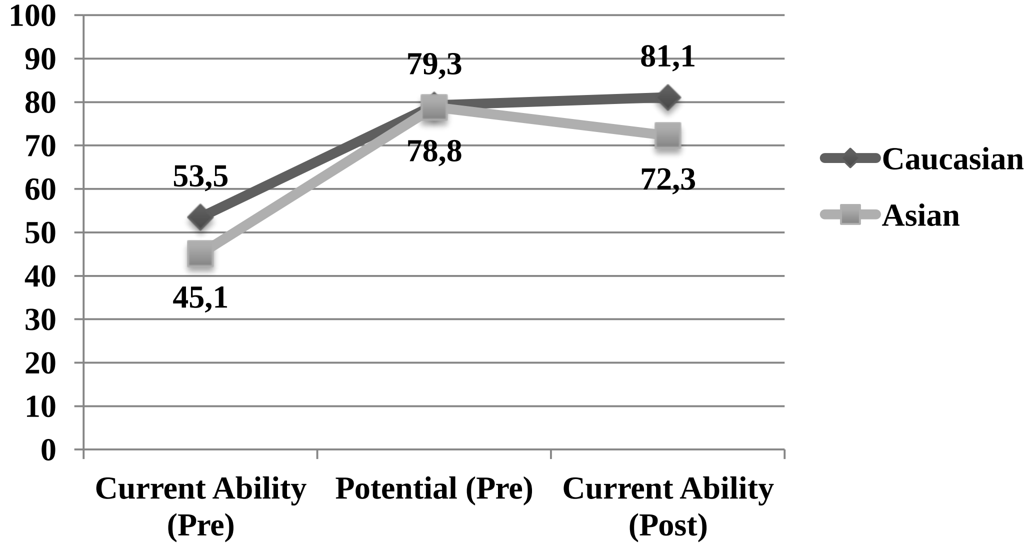

Figure 3. Pre and Post Self-Efficacy Comparisons by Race

As indicated in Table 2, students' self-efficacy on the pretest was lowest in the areas of creating examples that provide insight into designing proofs ($\bar{x}$= 30.5), engaging in intellectual arguments with others about mathematics ($\bar{x}$ = 34.4), constructing valid proofs ($\bar{x}$ = 36.3), and identifying the fallacious reasoning in incorrect proofs ($\bar{x}$ = 39.9). On the posttest, students rated their self-efficacy lowest in creating examples that provide insight into designing proofs ($\bar{x}$ = 68.9), identifying fallacious reasoning in incorrect proofs ($\bar{x}$ = 70.9), and working individually to solve mathematical problems from number theory and algebra ($\bar{x}$ = 71.8).

Table 2. Pretest and Posttest Self-Efficacy Means by Course Goal

		Pretest		Posttest
		Current Ability	Potential Ability	Current Ability
Understand the importance of proofs in mathematics.	$\bar{x}$	70.9	92.1	87.4
	SD	20.5	7.8	15.3
Use different methods to construct proofs.	$\bar{x}$	47.3	77.2	80.7
	SD	19.5	22.3	14.3
Understand the concept of "elegance" in proofs.	$\bar{x}$	52.2	78.8	80.2
	SD	28.5	23.5	19.8
Create examples that provide insight into designing proofs.	$\bar{x}$	30.5	74.7	68.9
	SD	20.6	19.0	17.2

Construct valid proofs.	$\bar{x}$	36.3	84.4	72.9
	SD	19.1	14.9	19.9
Identify fallacious reasoning in incorrect proofs.	$\bar{x}$	39.9	74.3	70.9
	SD	25.4	20.1	17.1
Engage in intellectual arguments with others about mathematics.	$\bar{x}$	34.4	72.2	72.8
	SD	28.2	27.0	21.0
Explain ideas that motivate proofs.	$\bar{x}$	48.0	74.4	78.4
	SD	21.4	18.4	16.6
Use concepts learned about elementary number theory and algebra to solve problems.	$\bar{x}$	53.3	80.8	78.5
	SD	29.5	21.4	20.0
Work individually to solve mathematical problems from number theory and algebra.	$\bar{x}$	46.1	81.3	71.8
	SD	30.0	19.4	28.5
Work in teams to solve mathematical problems from number theory and algebra.	$\bar{x}$	67.5	91.6	83.5
	SD	26.1	10.7	15.0
Define a "mathematical habit of mind."	$\bar{x}$	43.6	75.7	73.4
	SD	25.6	22.8	25.9

The areas in which students demonstrated the most positive changes in self-efficacy were creating examples that provide insight into designing proofs (post-pre difference = 38.4), engaging in intellectual arguments with others about mathematics (post-pre difference = 38.4), constructing valid proofs (post-pre difference = 36.7), using different methods to construct proofs (post-pre difference = 33.4), identifying the fallacious reasoning in incorrect proofs (post-pre difference = 31.00, and explaining ideas that motivate proofs (post-pre difference = 30.4).

Students overestimated their potential ability on 8 of the 12 course goals. The largest differences were in constructing valid proofs (12.4-point difference) and in working individually (9.5-point difference) and in teams to solve mathematical problems from number theory and algebra (8.1-point difference).

Students were asked on a post-course survey whether completing the course made them more confident about their mathematical ability or less confident about their mathematical ability. Eight students said *more confident*, 4 students said *less confident*, and 3 said *about the same*. Three of the 4 girls indicated they were more confident, and 1 reported being less confident. Five of the 11 boys said they were more confident, 3 said they were less confident, and 3 said their confidence was about the same.

When asked to describe the impact the course had on their confidence, 3 students commented that taking the course revealed to them how little they actually knew about mathematics, although these students had taken all the math courses offered at their high school and had completed college-level math course work. As one student explained,

> *Confidence generally relies heavily on the ratio of how much you perceive to know and how much you recognize you don't know. I believe I have learned a lot from this class; however, I have also learned what else is out there that I have never even heard about. Therefore, although my mathematical ability surely has increased, my confidence in the overall field of mathematics has somewhat decreased.*

A second student wrote, "This course opened up another 'branch' of mathematics that exposed how little I am able to do with math and how much I have not learned." A third student commented, "I know more about what mathematics is and ought to be, and I feel more comfortable with it."

Two other students described the difficulty of the course concepts and their belief that they could not solve problems without working with others. One student stated,

> *I really didn't understand most of the class. There was usually one integral piece of the puzzle needed to solve the problem. I didn't really have the mindset to come upon this piece most of the time without the help of others. I couldn't independently solve a lot of the problems.*

A student with a similar comment explained,

> *I feel less confident about my abilities in theoretical mathematics like this. I didn't find that I was able to easily understand all the concepts in the course. I don't think I can do much of this work by myself.*

A theme that came up repeatedly in student interviews related to the abstract nature of the course, and a number of students made comments about how different the PPNTA course was from the other math courses they had taken. For example, Aaron [all names are pseudonyms], in describing how his confidence had changed from the beginning to the end of the course, stated:

> *Well, I was kind of thinking, well, it's number theory, it's just about numbers, and I've been around numbers for many years, but then I realized it's more about the level of abstraction as you keep going along, so we learned about equivalence classes, so that was a new level of abstraction…or abstract algebra that I had no idea existed, and then we went into Group Theory... and that's even more abstraction, so I feel that aspect is a lot more intimidating than any, like, technical skills.*

Students frequently made comparisons between the "concrete" concepts they had covered in their other math classes and the abstract concepts covered in PPNTA. Following Aaron's comment, this exchange was made in the group interview:

> *Chen: There's no... like in calculus we could, if we're doing derivative, we could think of it in a concrete manner since it's just physics, but then, it's hard to try to come up…or you can't think of any kind of concrete examples when you're dealing with stuff like Group Theory.*
>
> *Chris: It's just like, you sit there and hope for it to come.*
>
> *SooJin: Yeah, it's like we have to let go of our preconceived notions, like what Aaron was saying about Group Theory, for example, well, multiplication doesn't exactly work anymore—the multiplication we're used to—so I guess it was a little hard to let go of what's "normal" to us.*
>
> *Vijay: But at the same time it's a lot cooler, because cryptography, that's just an application of the abstraction of equivalence classes, and I could see how math works at a higher level. It's that you abstract to... the fundamentals, and then you start applying, and then that leads to interesting results.*

Although all students who participated in this study earned an A as a final course grade and demonstrated their attainment of course objectives, as students in the different group interviews discussed their confidence in the course, it became clear that there was a group of students who "got" [an American idiom meaning to deeply understand something] the course and a group who didn't. Further, those groups tended to break down between students who were intrigued by studying abstract math and students who could not see the relevance of a course that, in their opinions, did not have a practical application. As Jin, who said she "didn't get" the course stated, "*The abstract kind of gets away from me a little bit...when I don't see the practical use of it*." In a different group interview, Amil explained:

> *I guess being someone who's more into application, if we do Number Theory and I can't see an immediate application to it, it doesn't really interest me that much, as opposed to differential equations or something that, we did word problems and stuff, and I was able to see that it was able to be applied to real-life situations.*

Those students who indicated they "didn't get" the PPNTA course often expressed their frustration at the abstract nature of the class and the lack of practical application. Jin explained:

> *It's an interesting course, but it's not my favorite class that I've ever had... it's just that it's something that I'd never actually see myself using in the future. It's interesting material to learn but it's nothing that I'd ever actually have to practice later... I think it really has to do a lot with how interested you are in math because, I find myself that I'm a*

> *lot more science focused than pure math based, so I like things that have a lot of meaning behind it and something that I can actually, that I know will be put into later use for me.*

But this attitude was in stark contrast to the students who did "get it" and were intrigued by the abstract nature of the class. Brady explained in his interview how taking the PPNTA course had reminded him how much he liked math, explaining that if he had taken the course earlier in his high school career, he might have considered minoring in math in college. He stated:

> *I wish I had taken [the class] sooner. I took so many calculus classes, I forgot how much I like math. Number Theory [PPNTA] was "the class," [that made me consider being a math major] because it's not an application class. Once you get past algebra—things you use in everyday life—you get to calculus, which is things you use in everyday life if you're an engineer. And Number Theory is just math for math. I really like that.*

Some of the students who felt they just didn't "get" the course had much lower confidence in their ability at the end of the course, and in fact, the four students who indicated on the post-survey that their confidence was lower also described, in interviews, how difficult it was for them to "get" PPNTA. This exchange occurred during a group interview with two male students, one who "got it" and had high confidence and one who didn't "get it" and had low confidence at the end of the course:

> *Amil: Speaking for myself and a few other people...it's pretty hard. You need a certain mindset to go into all the crazy abstract math and stuff, and it's hard for me.*
>
> *Brady: I'm one of the ones who likes it. There are clearly two groups of people in that class. There are people who get it and people who enjoy it and there are people who don't. It's kind of like and art class, even though it's math. There are people with an aptitude for it who walked in, able to do the stuff, and there are a lot of people who are working really hard and...*
>
> *Amil: who still don't get it.*

As these students continued to discuss they class, they described what it takes to "get it" in the course. Later in the interview, the students stated:

> *Brady: It's just something innate. Number theory does not look like a class that you can just work harder on. It's like, I can't draw. There's no amount of work I could put into an art class and do well in it. It's not an aptitude that I have, and I think [what's needed to get PPNTA] is a skill like that. It's not main track math, like algebra through calculus is a skill set, so if you practice harder you can amass the*

skills. But so much of Number Theory, like on our homework…so many of the problems depend on a piece of intuition.

Amil: Yeah.

Brady: There's usually one step in… half the problems on the homework will have one step that isn't something that we were ever taught. It isn't something that we've ever seen, it's just something that occurs to you. I had a problem on the last homework that I think I got right, and one of the steps basically needed me to do some weird inequality work like, we had two numbers that were between 0 and 1—or their absolute values were between 0 and 1—and I had to say, you know, if we multiply one of those by the whole set we're going to make it smaller because it's between 0 and 1. And it's not things we're learning in class. We're learning axioms, we're learning methods of proof. People are able to apply the skills they're picking up in class more or less, but the way the abstract math seems to be working, there's something else that's not being trained, if it's trainable.

Interviewer: Is that like [an] a-ha [moment]?

Both: Yeah.

Brady: It's an epiphany moment. It's funny. That's totally how…there are days I'll be working on it and I'll just be sitting there, and I'll have worked on a problem like 2 hours, essentially accomplishing nothing, and then I'll be like, "[Oh!] I know how to do this," and it's a 15-minute solution.

The other student described his lack of epiphany moments, which made it difficult for him to complete problems or do well in the course without help from his peers who "got it." Amil explained:

And I guess for me, I just like, don't get it, so there's no way I'm going to finish a problem, most of the time, until someone tells me how to do that one part... For the first midterm we had, I guess that stuff was simple enough that I was able to work on it by myself, but now that we're in group theory, which is a lot more abstract than just regular proofs and stuff, I think that I would just fail the class because I don't get it.

Brady responded to this:

See, now for me, the group work has its advantages for me, but a lot of how I function in there is just staring at something for long enough to just come up with an insight. So for me, I feel like I can arrive at an answer more quickly with a group of people to bounce ideas around, but I do well enough with the homework where it's just looking at a problem long enough to arrive at the insight by myself. But I'm not [like] the majority [of other students in the class] by a long shot.

In interviews, other students described how their self-efficacy was affected depending on what was being covered in the course, and for those students who had difficulty with abstract concepts, confidence was highest when topics were more concrete, as illustrated in this exchange:

> *Chen: [Confidence] depends on what we're doing, because we got to Group Theory, and I became a lot less confident, but before, like when we were doing cryptography... I got it, and I think our confidence was going up. Then we'd go into more abstract stuff, so it becomes less.*
>
> *Hema: Yeah, it's just like what we're learning affects or confidence [SooJin: yeah].*

In a different interview, Jin related:

> *I started off pretty confident when we started Group Theory because the basics—the closure, identity, the inverse—that was really concrete to me because you have to say that all these four are true in order for this to be a group. But then, after we did that and we started getting into all these different symbols, [the teacher] lost me, and I started getting really unsure about what I was doing, because it was just a lot of stuff that I didn't know.*

Fluctuation in self-efficacy was also described on the post course survey. As one student wrote:

> *It is true that I did learn a lot more and gain a lot more insight into the math field, so my confidence increased in that sense, but it took me an excessive amount of time to understand concepts. Sometimes, I would not even understand certain things, so that lowered my confidence. Overall, I am still at the same level of confidence.*

In some interviews, students related "getting" the course to the development of a mathematical habit of mind. Jin, for example, stated:

> *[A mathematical habit of mind is] being able to shift your perspective a little bit, and that's the kind of the thing that I lack... that I'm really focused on. [I'm] used to high school math where teachers tell you this is how you do a problem: this is the background with it, and that's all you need for these problems. But in Number Theory [PPNTA], it's a little more, you kind of have to be able to....you have to kind of forget a lot of things, like a lot of concrete details, and then you have to free your mind a little bit so you can accept new things, and I think that's a mathematical habit of mind. ...to accept different things that's the opposite of what you've been learning for the past 12 years.*

Aaron explained:

> *…it feels like whenever we do problems in Number Theory, it's exercising a new part of your brain, because we've been used to doing formulaic things, like just the algorithms in calculus. And then this is so abstract that you have to…it feels like you're developing something new, because you have to really think outside of what*

> *you've been trained to do...I guess that's leading to the mathematical habit of mind.*

For those students who "got" the course and whose self-efficacy increased as a result, there was much more interest in continuing to study abstract mathematics in college. As Aaron wrote on his survey,

> *Although I am not planning on becoming a mathematician, this course has made me more comfortable with the main aspects of being a mathematician: constructing and reviewing proofs, as well as handling abstractions… before I was not planning at all at taking any more math courses [in college], but now I feel comfortable taking ones in the future.*

As described earlier, Brady said in his interview that taking the PPNTA course had made him think more about taking advanced math courses in college. He stated,

> *I had ruled [being a math major]out, but I'm reevaluating that now. I was seeing myself as a physics major because I was afraid to be a math major. Then I took Number Theory [PPNTA] and remembered that I love math. So now I'm thinking about it again.*

Later in the interview he explained:

> *It's funny, right, but I wish I had started Number Theory [PPNTA] before I was applying to colleges, because it would have changed a lot of things. If they had been recorded, you could have looked at my college interviews before and after I started Number Theory and how much more excited I was in the later ones to say, "Hey, look at how cool math is. This is something I want to see again, even if it's not my major. Even if I minor in it, because I've declared myself as a physics major everywhere. So even if math isn't my major, I really love this and I want to keep pursuing it."*

Several other students explained that taking the PPNTA course had made them less interested in pursuing advanced math courses in college, giving reasons such as "I don't think I could do it in college," and "I'm pretty much not going to do any of this stuff ever again, and I don't get it anyway, so it's not like I'm going to do something extracurricular with it either." In her interview, Jin stated:

> *I'm kind of glad I took this class, because I was really thinking about maybe taking a number theory class later in college, but I'm glad I'm not doing it because I know it would go a lot faster than what's going on here… I'm glad I took it now instead of later...and I know that I don't want to be a math major.*

Another student explained that although participating in the PPNTA course had made him more interested in taking advanced math courses in college, it also had reduced his confidence that he could be successful in such courses.

CONCLUSIONS

Results of this study indicated that students' mathematical self-efficacy related to PPNTA course goals was low at the beginning of the course but was much higher at the end of the course. Additionally, when students were asked at the beginning of the course how confident they were that they could eventually achieve the course goals, their confidence was nearly identical to their post-course ratings. Though there was an increase in overall mathematics self-efficacy between the beginning and end of the course, 4 of the 15 participating students reported that their confidence had decreased during the course, and 3 students reported no change. Students who were less confident described their difficulty understanding the abstract mathematical concepts covered in the course. Some students also reported not having a full understanding of what mathematics is until being exposed to this course. That, however, impacted students in varying ways, making some more confident (particularly if they understood the concepts) but others less confident.

There was no indication that the adverse impact on some students' self-efficacy negatively affected their achievement in the course. All students earned an A as their final course grade, and students reported helping each other through the most difficult parts of the course. In future studies, collective efficacy, defined by Bandura (1997) as "a group's shared belief in its conjoint capabilities to organize and execute the courses of action required to produce given levels of attainment" (p. 477) should be examined, particularly if an emphasis on group problem-solving remains an important part of the course.

There was evidence that as some students' self-efficacy decreased, their persistence in achieving course goals also decreased, a view supported by Bandura's (1994, 1997) research. In fact, some of students who said they "didn't get" the course revealed in their interviews that they believed they did not have the innate ability needed to be successful in the course, particularly if they defined success as understanding the abstract concepts covered. Thus, even though they were succeeding based on the grading standards of the course (e.g., earning high grades), these students knew they did not understand many of the theoretical concepts covered and were not confident they could solve problems without significant help from their peers. These self-referent beliefs were much different than those held by their peers who did “get” and understand the theoretical concepts covered in the course and who were more apt to continue thinking about difficult problems while waiting for an epiphany or “a-ha” moment of insight. This result supports findings by Pajares (1996), whose research suggests that gifted students’ self-efficacy is affected more by their cognitive ability than by their performance. In this study, the students who “got it” were seen as having greater cognitive aptitude for abstract math. Students saw this aptitude as an ability to solve abstract math problems that required some type of innate ability related to intuition. In some cases, this intuition overlapped with students’ definitions of mathematical habits of mind, meaning that for some students, developing MHM may have been seen as beyond their capability.

There were several limitations in this study. First, the study was conducted with a small, non-random sample, which limits the generalizability of the results. Second, data were not collected on students’ self-regulation of learning, which makes it difficult to determine other factors that may have impacted students’ understanding, perseverance, and achievement in the course. We are currently replicating this study with additional groups of students to increase generalizability and are measuring students’ self-regulation strategies to determine whether those strategies differ for students with various levels of self-efficacy as well as for those students who have an affinity for the abstract, theoretical nature of the PPNTA course. Understanding the ways self-regulation differs between these groups of students may reveal ways to better facilitate learning in students with lower self-efficacy who believe they lack the capability to understand abstract, theoretical math concepts.

NOTE

This work is funded by Georgia's Race to the Top award sponsored by the U.S. Dept. of Education.

REFERENCES

Bandura, A. (1994). Self-efficacy. In V. S. Ramachaudran (Ed.), *Encyclopedia of human behavior* (Vol. 4, pp. 71-81). New York: Academic Press.

Bandura, A. (1997). *Self-efficacy: The exercise of control.* New York: Freeman.

Bandura, A. (2006). Guide for constructing self-efficacy scales. In T. Urdan & F. Pajares (Eds.), *Self-efficacy beliefs of adolescents* (pp. 207-337). Greenwich, CT: Information Age Publishing.

Cuoco, A., Goldenberg, E.P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15,* 375-402.

Millman, R., & Jacobbe, T. (2009). Mathematical habits of the mind in preservice teachers. *School Science and Mathematics*, *109*, 298-302.

Millman, R., & Jacobbe, T. (2008). Fostering creativity in preservice teachers through mathematical habit of the mind. In E. Velikova & A. Andž (Eds.), *Promoting creativity for all students in mathematics education*: *Proceedings of the Discussion Group 9* (pp. 158-163). Bulgaria: The 11th International Congress on Mathematical Education.

Multon, K., Brown, S., & Lent, R. (1991). Relation of self-efficacy beliefs to academic outcomes: A meta-analytic investigation. *Journal of Counseling Psychology, 38,* 30-38.

Pajares, F. (1996). Self-efficacy beliefs and mathematical problem-solving of gifted students. *Contemporary Educational Psychology, 21,* 325-344.

Pajares, F., & Graham, L. (1999). Self-efficacy, motivation constructs, and mathematics performance of entering middle school students. *Contemporary Educational Psychology, 24,* 124-139.

Pajares, F., &, J. (1995a). Self-efficacy beliefs and general mental ability in mathematics problem-solving. *Contemporary Educational Psychology, 20,* 426-443).

Pajares, F., & Miller, M.D. (1994). Role of self-efficacy and self-concept beliefs in mathematical problem-solving: A path analysis. *Journal of Educational Psychology, 86*(2), 193-203.

Pajares, F., & Schunk, D. (2001). Self-beliefs and school success: Self-efficacy, self-concept, and school achievement. In R. J. Riding & S. G. Rayner (Eds.), *International perspectives on individual differences* (Vol. 2): Self-perception (pp. 239-265). Westport, CT: Ablex.

Póyla, G. (1945). *How to solve it: A new aspect of mathematical method.* Princeton, NJ: Princeton University Press.

Zimmerman, B. J., Bandura, A., & Martinez-Pons, M. (1992). Self-motivation for academic attainments: The role of self-efficacy beliefs and personal goal setting. *American Educational Research Journal, 29,* 663-676

Submitted: March 2012

Accept: December 2012

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Appendix 1. Mathematics Self-Efficacy Pretest

The table below lists goals for the course in the MIDDLE COLUMN. Read the goal and then, in the LEFT COLUMN, mark how confident you are that you can already do this or have already reached that goal. In the RIGHT COLUMN, mark how confident you are that you can reach the goal. In each column, rate your degree of confidence by recording number from 0 to 100 using the scale given below:

0	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**
Cannot do at all					**Moderately can do**					**Highly certain can do**

Here's an example:

How confident are you that you can already do this?	Goal	How confident are you that you can learn to do this?
write in a number between 0 and 100		write in a number between 0 and 100
35	**Arrange a place to study without distractions.**	80

This means less than moderately confident one can already arrange a

This means between moderately and highly confident one can learn to arrange

How confident are you that you can already do this?	Goal	How confident are you that you can learn to do this?
write in a number between 0 and 100		write in a number between 0 and 100
	Understand the importance of proofs in mathematics.	
	Learn different methods to construct proofs.	
	Understand the concept of "elegance" in proofs.	
	Create examples that provide insight into designing proofs.	
	Construct valid proofs.	
	Identify the fallacious reasoning in incorrect proofs.	
	Engage in intellectual arguments with others about mathematics.	
	Explain ideas that motivate your proofs.	
	Use concepts learned about elementary number theory and algebra in other courses to solve problems in this course.	
	Work individually to solve mathematical problems from number theory and algebra.	
	Work in teams to solve mathematical problems from number theory and algebra.	
	Define what "mathematical habit of mind" means to you.	