MATH1061 Semester 2, 2002 Solution Ass. 3

Exercise Set 3.1

Q7 True as we only have to exhibit one example, and $5^2 = 3^2 + 4^2$.

Q26 Assume $n = 2k$ and $m = 2l + 1$, where $k, l \in \mathbb{Z}$. Then $n + m = 2k + 2l + 1 = 2(k + l) + 1$. Since $k + l \in \mathbb{Z}$ it follows that $n + m$ is odd as required.

Q41 This statement is false. For example let $m = 2$ and $n = 18$. Then $m.n = 36 = 6^2$, but m and n are not perfect squares.

Exercise Set 3.2

Q24 If $\dfrac{ax+b}{cx+d} = 1$ then $ax+b = cx+d$ and $ax - cx = d - b$. Implying $x(a-c) = d-b$ or $x = \dfrac{d-b}{a-c}$. Note $a \neq c$ so $a - c \neq 0$. And since $a, b, c, d \in \mathbb{Z}$ we have $a - c, d - b \in \mathbb{Z}$. Thus $x = \dfrac{p}{q}$ where $q = a - c \in \mathbb{Z}$ and $p = d - b \in \mathbb{Z}$. Thus x is rational.

Q28 Let $f(x) = r_0 + r_1 x + r_2 x^2 \ldots + r_n x^n$, where $r_i \in Q, 0 \leq i \leq n$. Then $r_i = \dfrac{p_i}{q_i}$ for $0 \leq i \leq n$ and $p_i, q_i \in \mathbb{Z}$. So

$$f(x) = \frac{p_0}{q_0} + \frac{p_1}{q_1}x + \frac{p_2}{q_2}x^2 + \ldots + \frac{p_n}{q_n}x^n.$$

Since c is a root of the polynomial $f(x)$ we have $f(c) = 0$. Hence

$$f(c) = \frac{p_0}{q_0} + \frac{p_1}{q_1}c + \frac{p_2}{q_2}c^2 + \cdots + \frac{p_n}{q_n}c^n = 0.$$

Multiplying throughout by $q_0 q_1 q_2 \ldots q_n$ we obtain

$$q_1 q_2 \ldots q_n p_0 + q_0 q_2 \ldots q_n p_1 c + q_0 q_1 q_3 \ldots q_n p_2 c^2 + \ldots + q_0 q_1 \ldots q_{n-1} p_n c^n = 0.$$

Thus $g(c) = 0$ where $g(x) = s_0 + s_1 x + \ldots + s_n x^n$, $s_i = q_0 \ldots q_{i-1} q_{i+1} \ldots q_n p_i$ and $s_i \in \mathbb{Z}$ as required.

Exercise Set 3.3

Q2 Yes, since $51 = 3 \times 17$

Q4 Yes, since $2m(2m + 2) = 4(m(m + 1))$

Q8 Yes, since $6a.10b = 2 \times 3a \times 2 \times 5b = 4(15ab)$

Q12 Yes. If $n - 1 = 4k + 3$ then

$$\begin{aligned} n^2 - 1 &= (4k+3)^2 - 1 \\ &= 16k^2 + 24k + 9 - 1 \\ &= 16k^2 + 24k + 8 \\ &= 8(2k^2 + 3k + 1) \end{aligned}$$

Q24 No. Let $a = 6, b = 9, c = 4$. Then $a|bc$ But $a \nmid b$ nor do we have $a \nmid c$.

Q36 a)

$$\begin{aligned} (20!)^2 &= (1 \times 2^1 \times 3^1 \times (2^2) \times 5^1 \times (2^1 \times 3^1) \times 7 \times (2^3) \times (3^2) \times (2^1 \times 5^1) \\ &\quad \times 11 \times (2^2 \times 3^1) \times 13 \times (2^1 \times 7) \times (3^1 \times 5^1) \times 2^4 \times 17 \times (2^1 \times 3^2) \\ &\quad \times 19 \times (2^2 \times 5^1))^2 \\ &= (2^{18} \times 3^8 \times 5^4 \times 7^2 \times 11 \times 13 \times 17 \times 19)^2 \\ &= 2^{36} \times 3^{16} \times 5^8 \times 7^4 \times 11^2 \times 13^2 \times 17^2 \times 19^2 \end{aligned}$$

b) $2 \times 5 = 10$. So $2^8 \times 5^8 = 10^8$. Hence $(20!)^2$ ends in 8 zeros.

Exercise Set 3.4

Q31 If n is not divisible by 2 or 3 it is of the form $6m + 1$ or $6m + 5$, where $m \in \mathbb{Z}$.

Case 1: $n = 6m + 1$

$$\begin{aligned} n^2 = (6m+1)^2 &= 36m^2 + 12m + 1 \\ &= 12(3m^2 + m) + 1 \end{aligned}$$

Since $3m^2 + m \in \mathbb{Z}$ we have $n^2 = 12k + 1$ where $k \in \mathbb{Z}$. Hence $n^2 \bmod 12 = 1$.

Case 2: $n = 6m + 5$

$$\begin{aligned} n^2 = (6m+5)^2 &= 36m^2 + 60m + 25 \\ &= 12(3m^2 + 5m + 2) + 1 \end{aligned}$$

Since $3m^2 + 5m + 2 \in \mathbb{Z}$ we have $n^2 = 12k + 1$ where $k \in \mathbb{Z}$. Hence $n^2 \bmod 12 = 1$.