MATH1061 Semester 2, 2002 Solutions Ass. 5

Exercise Set 4.2

Q7 Let $P(n)$ be the statement that $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$. That is, let $P(n)$ be the statement

$$\sum_{i=1}^{n} (4i - 3) = n(2n - 1).$$

Then when $n = 1$ we have $\sum_{i=1}^{1} (4i - 3) = 1$ and $1(2 \times 1 - 1) = 1$. Hence $P(1)$ is true.

Assume $P(k)$ is true. That is, assume $\sum_{i=1}^{k} (4i - 3) = k(2k - 1)$.

Prove $P(k+1)$ is true. That is, prove $\sum_{i=1}^{k+1} (4i - 3) = (k+1)(2(k+1) - 1)$.

$$\begin{aligned}
L.H.S = \sum_{i=1}^{k+1} (4i - 3) &= \sum_{i=1}^{k}(4i - 3) + 4(k+1) - 3 \\
&= k(2k - 1) + 4(k+1) - 3 \\
&= k(2k - 1) + 4(k+1) - 3 \\
&= 2k^2 - k + 4k + 4 - 3 \\
&= 2k^2 + 3k + 1. \\
R.H.S. = (k+1)(2(k+1) - 1) &= (k+1)(2k + 2 - 1) \\
&= (k+1)(2k+1) \\
&= 2k^2 + 3k + 1
\end{aligned}$$

Hence L.H.S. = R.H.S. and $P(k+1)$ is true. Thus by the Principle of Mathematical Induction $P(n)$ is true for all $n \geq 1$.

Q13 Let $P(n)$ be the statement $\sum_{i=1}^{n+1} i.2^i = n.2^{n+2} + 2$, $n \geq 0$.

Then when $n = 0$, $\sum_{i=1}^{0+1} i.2^i = 1.2^1 = 2$ and $0.2^{0+2} + 2 = 2$. So $P(0)$ is true.

Assume $P(k)$ is true. That is, assume $\sum_{i=1}^{k+1} i.2^i = k.2^{k+2} + 2$.

Prove $P(k+1)$ is true. That is, prove $\sum_{i=1}^{k+1+1} i2^i = \sum_{i=1}^{k+2} i2^i = (k+1)2^{(k+1)+2} + 2.$

$$\begin{aligned} L.H.S. = \sum_{i=1}^{k+2} i2^i &= \sum_{i=1}^{k+1} i2^i + (k+2)2^{k+2} \\ &= k2^{k+2} + 2 + (k+2)2^{k+2} \\ &= 2^{k+2}(k+k+2) + 2 \\ &= 2^{k+2}(2k+2) + 2 \\ &= 2^{k+2}.2(k+1) + 2 \\ &= 2^{k+3}(k+1) + 2 = R.H.S. \end{aligned}$$

Hence by the Principle of Mathematical Induction $P(n)$ is true, $\forall n \geq 0$.

Exercise Set 4.3

Q18 Let $P(n)$ be the statement that $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, $n \geq 2$. That is,

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}\,,\; n \geq 2.$$

When $n = 2$, $\sum_{i=1}^{2} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} > \frac{\sqrt{1}+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$. Hence $P(2)$ is true.

Assume $P(k)$ is true. That is, assume $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$.

Prove $P(k+1)$ is true. That is, prove $\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$.

$$\begin{aligned} \text{L.H.S.} = \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} &= \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \\ &> \sqrt{k} + \frac{1}{\sqrt{k+1}} \\ &= \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}} \\ &> \frac{\sqrt{k}\sqrt{k} + 1}{\sqrt{k+1}} \\ &= \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1} = R.H.S. \end{aligned}$$

Hence by the Principle of Mathematical Induction $P(n)$ is true, $\forall n \geq 2$.