THE UNIVERSITY OF QUEENSLAND
Department of Mathematics
MATH1061: Discrete Mathematics
Solution to the Second Exam, October 2002.
Exam Time: 50 minutes; Perusal Time: 5 minutes.

Question 1. By mathematical induction prove

$$\sum_{i=1}^{n+1} i.2^i = n.2^{n+2} + 2, \quad \text{for all integers} \quad n \geq 0.$$

(5 marks)

Solution Let $P(n)$ be the statement $\sum_{i=1}^{n+1} i.2^i = n.2^{n+2} + 2$, $n \geq 0$.

Then when $n = 0$, $\sum_{i=1}^{0+1} i.2^i = 1.2^1 = 2$ and $0.2^{0+2} + 2 = 2$. So $P(0)$ is true.

Assume $P(k)$ is true. That is, assume $\sum_{i=1}^{k+1} i.2^i = k.2^{k+2} + 2$.

Prove $P(k+1)$ is true. That is, prove $\sum_{i=1}^{k+1+1} i2^i = \sum_{i=1}^{k+2} i2^i = (k+1)2^{(k+1)+2} + 2$.

$$\begin{aligned} L.H.S. = \sum_{i=1}^{k+2} i2^i &= \sum_{i=1}^{k+1} i2^i + (k+2)2^{k+2} \\ &= k2^{k+2} + 2 + (k+2)2^{k+2} \\ &= 2^{k+2}(k+k+2) + 2 \\ &= 2^{k+2}(2k+2) + 2 \\ &= 2^{k+2}.2(k+1) + 2 \\ &= 2^{k+3}(k+1) + 2 = R.H.S. \end{aligned}$$

Hence by the Principle of Mathematical Induction $P(n)$ is true, $\forall n \geq 0$.

Question 2. By mathematical induction prove

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}, \quad \text{for all integers} \quad n \geq 1.$$

(5 marks)

Solution Let $P(n)$ be the statement

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}, \quad \text{for all integers} \quad n \geq 1.$$

Then when $n = 1$, $1^3 = \dfrac{1^2(1+1)^2}{4}$ and $1 = \dfrac{4}{4}$. So $P(1)$ is true.
Assume $P(k)$ is true. That is, assume

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}.$$

Prove $P(k+1)$ is true. That is, prove

$$1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}.$$

$$\begin{aligned}
\text{LHS} &= 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k^2+2k+1) + 4(k+1)^3}{4} \\
&= \frac{(k^4+2k^3+k^2) + (4k^3 + +12k^2 + 12k + 4)}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\text{RHS} &= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k^2+2k+1)(k^2+4k+4)}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}.
\end{aligned}$$

Hence LHS = RHD and now by the Principle of Mathematical Induction $P(n)$ is true, $\forall n \geq 1$.

Question 3. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Find the elements of $\mathcal{P}(A \times B)$. (Hint: First you need to find $A \times B$.) (5 marks)

Solution First note that $A \times B = \{(1,2), (1,3), (2,2), (2,3)\}$. So

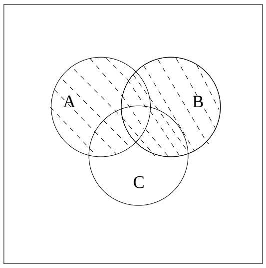
$$\begin{aligned}
\mathcal{P}(A \times B) = \; & \{\emptyset, \{(1,2)\}, \{(1,3)\}, \{(2,2)\}, \{(2,3)\}, \{(1,2),(1,3)\}, \\
& \{(1,2),(2,2)\}, \{(1,2),(2,3)\}, \{(1,3),(2,2)\}, \{(1,3),(2,3)\}, \\
& \{(2,2),(2,3)\}, \{(1,2),(1,3),(2,2)\}, \{(1,2),(1,3),(2,3)\}, \\
& \{(1,2),(2,2),(2,3)\}, \{(1,3),(2,2),(2,3)\}, \\
& \{(1,2),(1,3),(2,2),(2,3)\}\}.
\end{aligned}$$

Question 4. Use Venn diagrams to show that

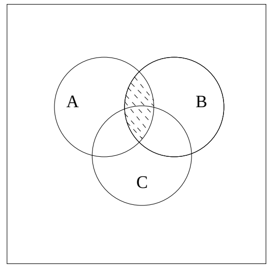
$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

(3 marks)

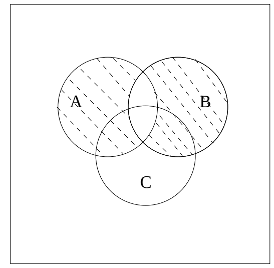
$A \cup B$



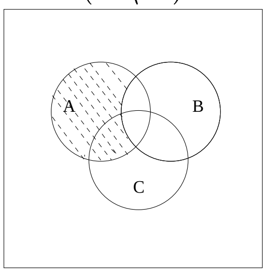
$A \cap B$



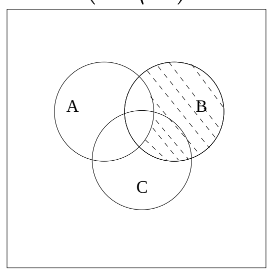
$(A \cup B) \setminus (A \cap B)$



$(A \setminus B)$



$(B \setminus A)$



$(A \setminus B) \cup (B \setminus A)$