LINEAR ALGEBRA

FIRST PART OF MATH2300

Table of Contents

1 BASICS OF LINEAR ALGEBRA

Quantum Physics, Mechanics, Wall Street and environmental management - what do all these things have in common? The answer, of course, is **Systems of Linear Equations**. That's right folks, whether you're looking at the Heisenberg Uncertainty Principle or controlling the amount of waste that pollutes our oceans you're going to be interested (of course you already are) in the study of Systems of linear equations. In this section some basic terminology will be introduced along with a method for solving such systems.

1.1 Basic Definitions

Definitions

- A **linear equation in n variables** $x_1, x_2, \ldots, x_n$ is one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

  where $a_1, a_2, \ldots, a_n$, and b are real constants. The variables in a linear equation are sometimes called the **unknowns**.

- An arbitrary **System of m linear equations in n unknowns** is one that can be written as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{1m}x_1 + a_{1m}x_2 + \cdots + a_{mn}x_n = b_m$$

  where once again $x_1, x_2, \cdots, x_n$ are the unknowns and the subscripted a's and b's denote constants.

- A system of linear equations is said to be **homogeneous** if the constant terms are all zero; that is, the system has the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{1m}x_1 + a_{1m}x_2 + \cdots + a_{mn}x_n = 0$$

- A sequence of numbers $s_1, s_2, \ldots, s_n$ is called a **solution** of the system if $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ is a solution of every equation in the system

Example
For example the below equations

$$4x_1 - x_2 + 3x_3 = -1$$
$$3x_1 + x_2 + 9x_3 = -4$$

are a system of 2 linear equations in 3 unknowns. The system has the solution $x_1 = 1, x_2 = 2, x_3 = -1$ since these values satisfy both equations. The set of values $x_1 = 1, x_2 = 8, x_3 = 1$ is not a solution since these values satisfy only the first of the two equations.

1.2 Solving a System of Linear Equations

1.2.1 The Augmented Matrix

Remembering that an arbitrary system of m linear equations in n unknowns can be written as:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{1m}x_1 + a_{1m}x_2 + \ldots + a_{mn}x_n = b_m$$

If one mentally keeps track of the +'s, the x's, and the ='s, then a system of m linear equations in n unknowns can be completely described by writing only the rectangular array of numbers:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Such a matrix is referred to as the **Augmented Matrix** of the system.

Remark

When constructing an augmented matrix, the unknowns must appear in the same order in each equation and the constants must be on the right.

1.2.2 Elementary Row Operations

As you know, the basic method for solving a system of linear equations is to replace the given system with a set that has the same solution set but is easier to solve. By applying the below three types of operation, unknowns are eliminated systematically without altering the solution set.

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a multiple of one equation to another.

Since the rows of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix. These operations are called **Elementary Row Operations**.

Definitions

- The three **Elementary Row Operations** are:
  1. Multiplying row i by a non-zero number t:
  $$R_i \to tR_i$$
  2. Interchange rows i and j:
  $$R_i \leftrightarrow R_j$$
  3. Adding t times row i to row j:
  $$R_j \to R_j + tR_i$$

- Matrix A is **row-equivalent** to matrix B if B is obtained from A by a sequence of elementary row operations. Clearly if B is row-equivalent to A then A is row-equivalent to B.

Remark
Elementary row operations can be used to convert an augmented matrix into a special form called reduced row-echelon form without changing the solution set. Once the conversion is complete the solution set is easily obtained.

1.2.3 Reduced Row-Echelon Form

Definitions

- A matrix is in **reduced row-echelon** form or **rref** for short when the following four conditions are satisfied:

  1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. (This is called a **leading 1.**)
  2. If there are any rows that consists entirely of zeros, then they are grouped together at the bottom of the matrix.
  3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
  4. Each column that contains a leading 1 has zeros everywhere else.

- A matrix having properties **1, 2** and **3** (but not necessarily 4) is said to be in **row echelon form** or **ref** for short.

Examples

- The following matrices are in reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The following matrices are in row-echelon form:

$$\begin{bmatrix} 1 & 4 & 3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1.2.4 Gauss Jordan Algorithm

Let A be the augmented matrix that completely describes a given system of m linear equations in n unknowns. It has already been stated that the solution set of this system can be easily obtained from the rref of A. Let rref of A be B. So the next question that needs to be asked is how do we convert A to B using elementary row operations. The answer can be found in the Gauss Jordan Algorithm.

The Gauss Jordan Algorithm converts a matrix A to its rref. The procedure is outlined below. As each step is stated, the idea will be illustrated by reducing the following matrix to reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

1. Locate the left most column that does not consist entirely of zeros. In the case above this is column one.

2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $\frac{1}{a}$ in order to introduce a leading 1.

$$R_1 \to \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$R_3 \to R_3 + -2R_1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

5. Now cover the top row in the matrix and begin again with step 1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row-echelon form. In this case our next leftmost nonzero column is column three.

$$R_2 \to \frac{-1}{2}R_2 \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$R_3 \to R_3 + -5R_2 \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

$$R_3 \to 2R_3 \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The entire matrix is now in row-echelon form. To find the reduced row-echelon form the following step is needed.

6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$R_2 \to R_2 + \frac{7}{2}R_3 \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1 \to R_1 + -6R_3 \begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1 \to R_1 + 5R_2 \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Remark

It can be shown that every matrix has a unique reduced row-echelon form.

1.3 A Sample Solution using Gauss-Jordan

Solve by Gauss-Jordan elimination:

$$\begin{array}{rrrrrrcr} x_1+ & 3x_2- & 2x_3 & & +\ 2x_5 & & = & 0 \\ 2x_1+ & 6x_2- & 5x_3- & 2x_4+ & 4x_5- & 3x_6 & = & -1 \\ & & 5x_3+ & 10x_4 & + & 15x_6 & = & 5 \\ 2x_1+ & 6x_2 & + & 8x_4+ & 4x_5+ & 18x_6 & = & 6 \end{array}$$

1. Form the corresponding Augmented Matrix for the system:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

2. Convert the Augmented Matrix to rref:

$$\begin{array}{l} R_2 \to R_2 + -2R_1 \\ R_4 \to R_4 + -2R_1 \end{array} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$R_2 \to -R_2 \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\begin{array}{l} R_3 \to R_3 + -5R_2 \\ R_4 \to R_4 + -4R_2 \end{array} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to \tfrac{1}{6}R_3 \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 + -3R_3 \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 + 2R_2 \quad \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Write down the corresponding system of equations:

$$\begin{aligned} x_1 + \ 3x_2 - \ 2x_3 \quad + \ 2x_5 \quad &= 0 \\ x_3 + \ 2x_4 \quad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

4. Solve each equation for the leading variables. In this case they are x_1, x_3 and x_6:
   $x_1 = -3x_2 - 4x_4 - 2x_5$
   $x_3 = -2x_4$
   $x_6 = \frac{1}{3}$

5. Assign the independent variables arbitrary values. In this case the independent variables are x_2, x_4 and x_5. Assign the arbitrary values r, s and t respectively. The general solution is given by the formulas.

$$x_1 = -3r - 4s - 2t, x_2 = r, x_3 = -2s, x_4 = s, x_5 = t, x_6 = \frac{1}{3}$$

1.4 Consistent and Inconsistent Systems

Definitions

- A system of linear equations that has no solution is said to be **inconsistent**.
- A system of linear equations with at least one solution is said to be **consistent**.

1.4.1 Demonstration of Inconsistent and Consistent Systems in the $x - y$ Plane

To illustrate the possibilities that can occur in solving systems of linear equations, consider a general system of two linear equations in the unknowns x and y:

$$a_1x + b_1y = c_1 \to l_1$$

$$a_2x + b_2y = c_2 \to l_2$$

(NB a_1 and b_1 not both zero and a_2 and b_2 not both zero). The graphs of these equations are lines: call them l_1 and l_2. Since a point (x, y) lies on a line if and only if the numbers x and y satisfy the equation of the line, the solutions of the system of equations correspond to points of intersection of l_1 and l_2. There are three possibilities:

- The lines l_1 and l_2 may be parallel, in which case there is no intersection and consequently no solution to the system. Thus the system of equations is said to be **inconsistent**.
- The lines l_1 and l_2 may intersect at only one point, in which case the system has exactly one solution. Thus the system of equations is said to be **consistent**.

- The lines l_1 and l_2 may coincide, in which case there are infinitely many points of intersection and consequently infinitely many solutions to the system. Thus this system is also said to be **consistent**.

Although we have only demonstrated this for a special case, it can be shown in general that the following theorem holds.

Theorem 1.1 *Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.*

1.5 Solving Homogeneous Systems of Linear Equations

Solving the below homogeneous system of linear equations

$$\begin{array}{rrrrrl} 2x_1+ & 2x_2- & x_3 & & +\ \ x_5 = & 0 \\ -x_1- & x_2+ & 2x_3- & 3x_4+ & x_5 = & 0 \\ x_1+ & x_2- & 2x_3 & & -\ \ x_5 = & 0 \\ & & x_3+ & x_4+ & x_5 = & 0 \end{array}$$

The augmented matrix for the system is

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Reducing the matrix to rref, we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{array}{rrrrrl} x_1+ & x_2 & & & +\ \ x_5 = & 0 \\ & & x_3 & & +\ \ x_5 = & 0 \\ & & & x_4 & & 0 \end{array}$$

Solve each equation for the leading variables
$x_1 = -x_2 - x_5$
$x_3 = -x_5$
$x_4 = 0$

Then the general solution is

$$x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$$

Note that the trivial solution is obtained when $s = t = 0$.

This example illustrates two important points about solving a homogeneous system of linear equations:

- None of the three elementary row operations alter the final column of zero's. Therefore, the corresponding system of equations to the rref is also a homogeneous system of linear equations.

- Depending on the number of zero rows in the rref the number of equations in the reduced system is less than or equal to the number of equations in the original system.

These two observations are instrumental in proving the following theorem.

Theorem 1.2 *A Homogeneous System of linear equations with more unknowns than equations has infinitely many solutions*

Proof: If a given homogeneous system has m equations; in n unknowns such that $m < n$, and if there are r non-zero rows in the rref of the augmented matrix, it follows that $r < n$. The system of equations corresponding to the rref of the augmented matrix will have the following form:

$$\begin{array}{llllll} \cdots x_{k_1} & & & & +\sum(\,) & = 0 \\ & \cdots x_{k_2} & & & +\sum(\,) & = 0 \\ & & \cdots \ddots & & \vdots & \\ & & & x_{k_r} & +\sum(\,) & = 0 \end{array}$$

where $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ are the leading variables and $\sum(\,)$ denotes sums (possibly all different) that involve the $n - r$ free variables. Solving for the leading variables gives

$$\begin{aligned} x_{k_1} &= -\sum(\,) \\ x_{k_2} &= -\sum(\,) \\ &\vdots \\ x_{k_r} &= -\sum(\,) \end{aligned}$$

As in the above example, arbitrary values can be assigned to the free variables on the right hand side and thus infinitely many solutions are obtained for the system.

1.6 Important Matrices

There are two matrices which are of special significance in linear algebra:
Definitions

- Square matrices taking the form of 1's on the main diagonal and 0's off the main diagonal, are called **Identity Matrices**. Such matrices are denoted by I. If it is important to emphasize the size, I_n shall be written to denote the $n \times n$ identity matrix. For instance I_3 is the matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  If A is an $m \times n$ matrix, then

$$AI_n = A \ \ and \ \ I_m A = A$$

  Thus an identity matrix plays much the same role in matrix arithmetic as the number 1 plays in the numerical relationship $a \cdot 1 = 1 \cdot a = 1$.

- If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be **invertible** or **non-singular** and B is called the **inverse** of A. If A is not invertible it is said to be **singular**.

- A square matrix in which all of the entries off the main diagonal are zero is called a **diagonal matrix**. Some examples are:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

A general $n \times n$ matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

1.6.1 Properties of Diagonal Matrices

Diagonal matrices enjoy the following properties.

- A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of the general form is

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \dots & 0 \\ 0 & \frac{1}{d_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{d_n} \end{bmatrix}$$

Verify that $DD^{-1} = D^{-1}D = I$.

- Powers of diagonal matrices are easy to compute; once again it is left to the reader to verify that if D is a the general diagonal matrix and k is any positive integer, then

$$D^k = \begin{bmatrix} {d_1}^k & 0 & \dots & 0 \\ 0 & {d_2}^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & {d_n}^k \end{bmatrix}$$

- A diagonal matrix is its own transpose. That is to say that $D = D^T$.

- To multiply a matrix A on the left by a diagonal matrix D, one can multiply successive rows of A by the successive diagonal entries of D. For example:

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

- To multiply a matrix A on the right by a diagonal matrix D, one can multiply successive rows of A by the successive diagonal entries of D. For example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

2 VECTOR SPACES AND SUBSPACES

What is a vector? Many are familiar with the concept of a vector as:

- Something which has magnitude and direction.
- an ordered pair or triple.
- a description for such quantities as Force, velocity and acceleration.

Such vectors belong to the foundation vector space - $\Re^n$ - of all vector spaces. The properties of general vector spaces are based on the properties of $\Re^n$. It is therefore helpful to consider briefly the nature of $\Re^n$.

2.1 The Vector Space $\Re^n$

Definitions

- If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is called **n-space** and is denoted by $\Re^n$.

When $n = 1$ each ordered n-tuple consists of one real number, and so $\Re$ may be viewed as the set of real numbers. Take $n = 2$ and one has the set of all 2-tuples which are more commonly known as **ordered pairs**. This set has the geometrical interpretation of describing all points and directed line segments in the Cartesian $x - y$ plane. The vector space $\Re^3$, likewise is the set of **ordered triples**, which describe all points and directed line segments in 3-D space.

In the study of 3-space, the symbol (a_1, a_2, a_3) has two different geometric interpretations: it can be interpreted as a point, in which case a_1, a_2 and a_3 are the coordinates, or it can be interpreted as a vector, in which case a_1, a_2 and a_3 are the components. It follows, therefore, that an ordered n-tuple $(a_1, a_2, \ldots, a_n)$ can be viewed as a "generalized point" or a "generalized vector" - the distinction is mathematically unimportant. Thus, we can describe the 5-tuple $(1, 2, 3, 4, 5)$ either as a point or a vector in $\Re^5$.

Definitions

- Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\Re^n$ are called **equal** if

$$u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$$

- The **sum** $u + v$ is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

- Let k be any scalar, then the **scalar multiple** $k\mathbf{u}$ is defined by

$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

- These two operations of addition and scalar multiplication are called the **standard operations** on $\Re^n$.
- The **zero vector** in $\Re^n$ is denoted by $\mathbf{0}$ and is defined to be the vector

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- The **negative** (or **additive inverse**) of u is denoted by -u and is defined by

$$-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$$

- The **difference** of vectors in $\Re^n$ is defined by

$$\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u})$$

The most important arithmetic properties of addition and scalar multiplication of vectors in $\Re^n$ are listed in the following theorem. This theorem enables us to manipulate vectors in $\Re^n$ without expressing the vectors in terms of components.

Theorem 2.1 *If* $\mathbf{u} = (u_1, u_2, \ldots, u_n), \mathbf{v} = (v_1, v_2, \ldots, v_n)$*, and* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ *are vectors in* $\Re^n$ *and* k *and* l *are scalars, then:*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$*; that is,* $\mathbf{u} - \mathbf{u} = \mathbf{0}$
5. $k(l\mathbf{u}) = (kl)\mathbf{u}$
6. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
7. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

2.2 Generalized Vector Spaces

The time has now come to generalize the concept of a vector. In this section a set of axioms are stated, which if satisfied by a class of objects, entitles those objects to be called "vectors." The axioms were chosen by abstracting the most important properties (Theorem 2.1) of vectors in $\Re^n$; as a consequence, vectors in $\Re^n$ automatically satisfy these axioms. Thus, the new concept of a vector, includes many new kinds of vector without excluding the 'common vector'. The new types of vectors include, among other things, various kinds of matrices and function. (But how on earth can a function be a frigging vector? Trust me my faithful readers, it will all become clear). The work covered in this section provides a powerful tool for the extension of geometrical visualisation to a wide variety of important mathematical problems where geometric intuition is otherwise unavailable. Briefly stated the concept is this: It is possible to visualize vectors in $\Re^2$ and $\Re^3$ geometrically as arrows, which enables the drawing of physical and mental pictures which aid the solution of the problem. Now the axioms used to create the new kinds of vectors are based on the properties of vectors in $\Re^2$ and $\Re^3$, and therefore have many of the familiar properties of vectors in $\Re^2$ and $\Re^3$. Consequently, the solving of problems involving the new kinds of vectors, say matrices or functions, may be aided by visualizing geometrically what the corresponding problem would be like in $\Re^2$ or $\Re^3$.

Definitions

- Let V be an arbitrary nonempty set of objects on which two operations are defined, addition and multiplication by scalars (number). **Addition** is a rule (not necessarily the standard rule) for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u} + \mathbf{v}$, called the **sum** of $\mathbf{u}$ and $\mathbf{v}$; **scalar multiplication** is a rule (not necessarily the standard rule) for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the **scalar multiple** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and l, then V is called a **vector space** and the objects in V are called **vectors**.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a **zero vector** for V, such that $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object -$\mathbf{u}$ in V, called the **negative** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$;
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(l\mathbf{u})=(kl)\mathbf{u}$
8. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
9. $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$
10. $1\mathbf{u}=\mathbf{u}$

Remark Depending on the application, scalars may be either real numbers or complex numbers. Vector spaces in which the scalars are complex are referred to as Complex vector spaces. Vector spaces in which the scalars must by Real are referred to as Real vector Spaces. Discussion in this subject will be limited to Real Vector Spaces.

It is of the utmost importance to keep in mind that the definition of a vector space specifies neither the nature of the vectors nor the operations. It is possible for any kind of object to be a vector, and the operations of addition and scalar multiplication may not have any relationship or similarity to the standard vector operations on $\Re^n$. The only requirement is that the ten vector space axioms be satisfied. The notations $\oplus$ and $\odot$ are used in the notes for vector addition and scalar multiplication to distinguish between these operations and the standard vector operations previously introduced.

Examples of Vector Spaces

A wide variety of Vector spaces are possible under the above definition as illustrated by the following examples. In each example a nonempty set of V and two operations: addition and scalar multiplication will be specified. Then it shall be demonstrated that the 10 axioms are satisfied, hence entitling V with the specified operations, to be called a vector space.

1. Show that the set V for all 2×2 matrices with real entries is a vector space if vector addition is defined to be matrix addition and vector scalar multiplication is defined to be matrix scalar multiplication.

   In this example the axioms will be verified in the following order: 1, 6, 2, 3, 4, 5, 7, 8, 9 and 10. Let

   $$\mathbf{u}=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \ \mathit{and}\ \mathbf{v}=\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

   To prove Axiom 1, it must be shown that $\mathbf{u}+\mathbf{v}$ is an object in V; that is, it must be shown that $\mathbf{u}+\mathbf{v}$ is a 2×2 matrix. But this follows from the definition of matrix addition, since

   $$\mathbf{u}+\mathbf{v}=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

   Similarly, Axiom 6 holds because for any real number k

   $$k\mathbf{u}=k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

   so that $k\mathbf{u}$ is a 2×2 matrix and consequently is an object in V.

Proving Axiom 2:

$$\mathbf{u}+\mathbf{v}=\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]+\left[\begin{array}{cc} v_{11} & v_{12} \\ v_{21} & v_{22} \end{array}\right]=\left[\begin{array}{cc} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{array}\right]$$

$$=\left[\begin{array}{cc} v_{11}+u_{11} & v_{12}+u_{12} \\ v_{21}+u_{21} & v_{22}+u_{22} \end{array}\right]=\left[\begin{array}{cc} v_{11} & v_{12} \\ v_{21} & v_{22} \end{array}\right]+\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]=\mathbf{v}+\mathbf{u}$$

Proving axiom 3:

$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]+\left(\left[\begin{array}{cc} v_{11} & v_{12} \\ v_{21} & v_{22} \end{array}\right]+\left[\begin{array}{cc} w_{11} & w_{12} \\ w_{21} & w_{22} \end{array}\right]\right)$$

$$=\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]+\left[\begin{array}{cc} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22} \end{array}\right]$$

$$=\left[\begin{array}{cc} u_{11}+v_{11}+w_{11} & u_{12}+v_{12}+w_{12} \\ u_{21}+v_{21}+w_{21} & u_{22}+v_{22}+w_{22} \end{array}\right]$$

$$=\left[\begin{array}{cc} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{array}\right]+\left[\begin{array}{cc} w_{11} & w_{12} \\ w_{21} & w_{22} \end{array}\right]$$

$$=\left(\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]+\left[\begin{array}{cc} v_{11} & v_{12} \\ v_{21} & v_{22} \end{array}\right]\right)+\left[\begin{array}{cc} w_{11} & w_{12} \\ w_{21} & w_{22} \end{array}\right]=(\mathbf{u}+\mathbf{v})+\mathbf{w}$$

To prove axiom 4, an object $\mathbf{0}$ must be found in V such that $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{0}$ for all $\mathbf{u}$ in V. Define $\mathbf{0}$ to be

$$\mathbf{0}=\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$$

With this definition

$$\mathbf{0}+\mathbf{u}=\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]+\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]=\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]=\mathbf{u}$$

and similarly $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

To prove axiom 5, it must be shown that for every object $\mathbf{u}$ in V there is a negative -$\mathbf{u}$ such that $\mathbf{u}+-\mathbf{u}=\mathbf{0}$. Let the negative of $\mathbf{u}$ be

$$-\mathbf{u}=\left[\begin{array}{cc} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{array}\right]$$

With this definition

$$\mathbf{u}+(-\mathbf{u})=\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]+\left[\begin{array}{cc} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{array}\right]=\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]=\mathbf{0}$$

and similarly $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.

Proving axiom 7:

$$k(l\mathbf{u})=k\left(l\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]\right)=k\left[\begin{array}{cc} lu_{11} & lu_{12} \\ lu_{21} & lu_{22} \end{array}\right]=\left[\begin{array}{cc} klu_{11} & klu_{12} \\ klu_{21} & klu_{22} \end{array}\right]$$

$$=kl\left[\begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array}\right]=(kl)\mathbf{u}$$

Proving axiom 8:

$$k(\mathbf{u}+\mathbf{v}) = k\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}\right) = k\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11}+kv_{11} & ku_{12}+kv_{12} \\ ku_{21}+kv_{21} & ku_{22}+kv_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} kv_{11} & kv_{12} \\ kv_{21} & kv_{22} \end{bmatrix}$$

$$= k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + k\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = k\mathbf{u} + k\mathbf{v}$$

Proving axiom 9:

$$(k+l)\mathbf{u} = (k+l)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} (k+l)u_{11} & (k+l)u_{12} \\ (k+l)u_{21} & (k+l)u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11}+lu_{11} & ku_{12}+lu_{12} \\ ku_{21}+lu_{21} & ku_{22}+lu_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} lu_{11} & lu_{12} \\ lu_{21} & lu_{22} \end{bmatrix} = k\mathbf{u} + l\mathbf{u}$$

Axiom 10 is a simple computation

$$1\mathbf{u} = 1\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

Therefore the set of all real 2×2 matrices, with matrix addition and matrix scalar multiplication form a Vector Space

2. Let V by the set of real-valued functions defined on the entire real line $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two such functions and k is any real number, define the sum function $\mathbf{f}+\mathbf{g}$ and the scalar multiple $k\mathbf{f}$ by

$$(\mathbf{f}+\mathbf{g})(x) = f(x) + g(x)$$
$$(k\mathbf{f})(x) = kf(x)$$

It is again convenient to verify the axioms in the following order: 1, 6, 2, 3, 4, 5, 7, 8, 9 and 10.

To prove axiom 1 it must be shown that $\mathbf{f}+\mathbf{g}$ is an object in V; that is it must be shown that $\mathbf{f}+\mathbf{g}$ is a real-valued function defined on the entire real line. This is quite simple.

$$f(x) \in \Re \quad \forall \quad x \in (-\infty, \infty)$$

$$g(x) \in \Re \quad \forall \quad x \in (-\infty, \infty)$$

therefore

$$(\mathbf{f}+\mathbf{g})(x) = f(x) + g(x) \in \Re \quad \forall \quad x \in (-\infty, \infty)$$

In the same way axiom 6 holds because for any real number k we have

$$(k\mathbf{f})(x) = kf(x) \in \Re \quad \forall \quad x \in (-\infty, \infty)$$

Proving Axiom 2:

$$(\mathbf{f}+\mathbf{g})(x) = f(x) + g(x) = g(x) + f(x) = (\mathbf{g}+\mathbf{f})(x)$$

$$\forall \ x \in (-\infty, \infty)$$

Proving Axiom 3:

$$(\mathbf{f} + (\mathbf{g} + \mathbf{h}))(x) = f(x) + (g(x) + h(x)) = f(x) + g(x) + h(x)$$
$$= (f(x) + g(x)) + h(x) = ((\mathbf{f} + \mathbf{g}) + \mathbf{h})(x)$$
$$\forall \ x \in (-\infty, \infty)$$

Proving Axiom 4:

Let the **0** vector be the constant function that is identically zero for all values of x. The graph of this function is the line that coincides with the x-axis. Then quite clearly

$$(\mathbf{f} + \mathbf{0})(x) = f(x) + 0 = (\mathbf{0} + f)(x) = \mathbf{f}$$
$$\forall \ x \in (-\infty, \infty)$$

Proving Axiom 5:

Let the negative of **f** be -**f** = -f(x). Then quite clearly

$$(\mathbf{f} + (-\mathbf{f}))(x) = f(x) + -f(x) = \mathbf{0}$$
$$\forall \ x \in (-\infty, \infty)$$

Proving Axiom 7:

$$(k + l)\mathbf{f} = (k + l)f(x) = kf(x) + lf(x) = (kf)(x) + (kl)(x) = k\mathbf{f} + l\mathbf{f}$$
$$\forall \ x \in (-\infty, \infty)$$

Proving Axiom 8:

$$k(l\mathbf{f}) = k(lf(x)) = klf(x) = (kl)f(x) = (kl)\mathbf{f}$$
$$\forall \ x \in (-\infty, \infty)$$

Proving Axiom 9:

$$(k(\mathbf{f} + \mathbf{g}))(x) = k(f(x) + g(x)) = kf(x) + kg(x) = (k\mathbf{f})(x) + (k\mathbf{g})(x)$$
$$\forall \ x \in (-\infty, \infty)$$

Proving Axiom 10:

$$\mathbf{1f} = 1f(x) = f(x) = \mathbf{f}$$
$$\forall \ x \in (-\infty, \infty)$$

3. Let $V = \Re^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

For example, if $\mathbf{u} = (2, 3), \mathbf{v} = (5, 2), k = 4$, then

$$\mathbf{u} + \mathbf{v} = (2 + 5, 3 + 2) = (7, 5)$$

$$k\mathbf{u} = 4\mathbf{u} = (4 \cdot 3, 0) = (12, 0)$$

Note that the addition operation is the standard addition operation on $\Re^2$, but the scalar multiplication operation is not standard. It can be shown that the first nine vector axioms hold. However there are values of $\mathbf{u}$ for which Axiom 10 fails to hold. For example, if $\mathbf{u} = (u_1, u_2)$ such that $u_2 \neq 0$, then

$$1\mathbf{u} = 1(u_1, u_2) = (1 \cdot u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

Thus, V is not a vector space with the stated operations.

2.2.1 Some Properties of Vectors

More and more examples of vector spaces will be added to the list as the course continues. It is important to realize that all vector spaces obey the following theorem which lists a useful set of vector properties.

Theorem 2.2 *Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:*

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = 0$.

This theorem is easily proved by using the 10 axioms of Vector Spaces.

2.3 Subspaces

It is possible for one vector space to be contained within a larger vector space. This section will look closely at this important concept.
Definitions

- A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In general, all ten vector space axioms must be verified to show that a set W with addition and scalar multiplication forms a vector space. However, if W is part of a largest set V that is already known to be a vector space, then certain axioms need not be verified for W because they are "inherited" from V. For example, there is no need to check that $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ (Axiom 2) for W because this holds for all vectors in V and consequently holds for all vectors in W. Likewise, Axioms 3, 7, 8, 9 and 10 are inherited by W from V. Thus to show that W is a subspace of a vector space V (and hence that W is a vector space), only Axioms 1, 4, 5 and 6 need to be verified. The following theorem reduces this list even further by showing that even Axioms 4 and 5 can be dispensed with.

Theorem 2.3 *If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.*

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

(b) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.

Proof: If W is a subspace of V, then all the vector space axioms are satisfied; in particular, Axioms 1 and 6 hold. But these are precisely conditions (a) and (b).

Conversely, assume conditions (a) and (b) hold. Since these conditions are vector space Axioms 1 and 6, it only remains to be shown that W satisfies the remaining eight axioms. Axioms 2, 3, 7, 8, 9 and 10 are automatically satisfied by the vectors in W since they are satisfied by all vectors in V. Therefore, to complete the proof, we need only verify that Axioms 4 and 5 are satisfied by vectors in W.

Let $\mathbf{u}$ be any vector in W. By condition (b), $k\mathbf{u}$ is in W for every scalar k. Setting $k = 0$, it follows from Theorem 2.2 that $0\mathbf{u} = \mathbf{0}$ is in W, and setting $k = -1$, it follows that $(-1)\mathbf{u} = -\mathbf{u}$ is in W.

Remarks

- Note that a consequence of (b) is that $\mathbf{0}$ is an element of W.
- A set W of one or more vectors from a vector space V is said to be **closed under addition** if condition (a) in Theorem 2.3 holds and **closed under scalar multiplication** if condition (b) holds. Thus, Theorem 2.3 states that W is *a* subspace of V if and only if W is closed under addition and closed under scalar multiplication.

Examples of Subspaces

1. A plane through the origin of $\Re^3$ forms a subspace of $\Re^3$. This is evident geometrically as follows: Let W be any plane through the origin and let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in W other than the zero vector. Then $\mathbf{u}+\mathbf{v}$ must lie in W because it is the diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, and $k\mathbf{u}$ must lie in W for any scalar k because $k\mathbf{u}$ lies on a line through $\mathbf{u}$. Thus, W is closed under addition and scalar multiplication, so it is a subspace of $\Re^3$.

2. A line through the origin of $\Re^3$ is also a subspace of $\Re^3$. It is evident geometrically that the sum of two vectors on this line also lies on the line and that a scalar multiple of a vector on the line is on the line as well. Thus, W is closed under addition and scalar multiplication, so it is a subspace of $\Re^3$.

3. Let n be a positive integer, and let W consist of all functions expressible in the form

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

where $a_0, \ldots, a_n$ are real numbers. Thus, W consists of the zero function together with all real polynomials of degree n or less. The set W is a subspace of the vector space of all real-valued functions discussed in Example 2 of section 2.2. To see this, let $\mathbf{p}$ and $\mathbf{q}$ be the polynomials

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

and

$$q(x) = b_0 + b_1 x + \ldots + b_n x^n$$

Then

$$(\mathbf{p} + \mathbf{q})(x) = p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$$

and

$$(k\mathbf{p})(x) = kp(x) = (ka_0) + (ka_1)x + \ldots + (ka_n)x^n$$

These functions have the form given above, so $\mathbf{p} + \mathbf{q}$ and $k\mathbf{p}$ lie in W. This vector space W shall be denoted by the symbol P_n.

2.4 Linear Combination of Vectors

Definitions

- A vector $\mathbf{w}$ is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_r$ if it can be expressed in the form
$$\mathbf{w} = k_1 v_1 + k_2 v_2 + \ldots + k_r v_r$$
where $k_1, k_2, \ldots, k_r$ are scalars.

Example

1. Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\Re^3$. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

   In order for $\mathbf{w}$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $k_1\mathbf{u} + k_2\mathbf{v} = \mathbf{w}$; that is in matrix form,
$$\begin{bmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$$
   Forming the augmented matrix of the system gives
$$\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix}$$
   Finding the rref of the augmented matrix gives
$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
   The corresponding equations simply state $k_1 = -3$ and $k_2 = 1$. Therefore $\mathbf{w} = -3\mathbf{u} + 1\mathbf{v}$

   Similarly, for **w'** to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$; that is in matrix form,
$$\begin{bmatrix} 1 & 6 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$$
   Forming the augmented matrix of the system gives
$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix}$$
   Finding the rref of the augmented matrix gives
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
   Clearly from the last row of the rref the system is inconsistent and therefore, no such scalars k_1 or k_2 exist. Consequently $\mathbf{w}'$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

2.4.1 Spanning

If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ are vectors in a vector space V, then generally some vectors in V may be linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ and others may not. The following theorem shows that if a set W is constructed consisting of all those vectors that are expressible as linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$, then W forms a subspace of V.

Theorem 2.4 *If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ are vectors in a vector space V, then:*

(a) The set W of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ is a subspace of V.

(b) W is the smallest subspace of V that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ every other subspace of V that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ must contain W

Proof:

(a) To show that W is a subspace of V, it must be proven that it is closed under addition and scalar multiplication. There is at least one vector in W, namely, $\mathbf{0}$, since $\mathbf{0} = 0\mathbf{v_1} + 0\mathbf{v_2} + \ldots + 0\mathbf{v_r}$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then

$$\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_r\mathbf{v_r}$$

and

$$\mathbf{v} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \ldots + k_r\mathbf{v_r}$$

where $c_1, c_2, \ldots, c_r, k_1, k_2, \ldots, k_r$ are scalars. Therefore

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{v_1} + (c_2 + k_2)\mathbf{v_2} + \ldots + (c_r + k_r)\mathbf{v_r}$$

and, for any scalar k,

$$k\mathbf{u} = (kc_1)\mathbf{v_1} + (kc_2)\mathbf{v_2} + \ldots + (kc_r)\mathbf{v_r}$$

Thus, $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ and consequently lie in W. Therefore, W is closed under addition and scalar multiplication.

(b) Each vector $\mathbf{v_i}$ is a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ since we can write

$$\mathbf{v_i} = 0\mathbf{v_1} + 0\mathbf{v_2} + \ldots + 1\mathbf{v_i} + \ldots + 0\mathbf{v_r}$$

Therefore, the subspace of W contains each of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Let W' be any other subspace that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Since W' is closed under addition and scalar multiplication, it must contain all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Thus W' contains each vector of W.

Definitions

- If S $= \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of all linear combinations of the vectors in S is called the **space spanned** by $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$, and it is said that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ **span** W. To indicate that W is the space spanned by the vectors in the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ the below notation is used.

$$W = span(S) \quad or \quad W = span\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$$

Examples The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n defined previously since each polynomial $\mathbf{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n$$

which is a linear combination of $1, x, x^2, \dots, x^n$. This can be denoted by writing

$$P_n = span\{1, x, x^2, \dots, x^n\}$$

Spanning sets are not unique. For example, any two noncollinear vectors that lie in the $x - y$ plane will span the $x - y$ plane. Also, any nonzero vector on a line will span the same line.

Theorem 2.5 *If $S = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_r}\}$ and $S' = \{\mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w_k}\}$ are two sets of vectors in a vector space V, then*

$$span\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_r}\} = span\{\mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w_k}\}$$

if and only if each vector in S is a linear combination of those in S', and conversely each vector in S' is a linear combination of those in S.

Proof: If each vector in S is a linear combination of those in S' then

$$span(S) \in span(S')$$

If each vector in S' is a linear combination of those in S then

$$span(S') \in span(S)$$

and therefore

$$span(S) = span(S')$$

if

$$\mathbf{v_i} \neq a_1\mathbf{w_1} + a_2\mathbf{w_2} + \cdots + a_n\mathbf{w_n}$$

for all possible $a_1, a_2, \dots, a_n$ then

$$\mathbf{v_i} \in span(S) \quad but \quad \mathbf{v_i} \notin span(S')$$

therefore

$$span(S) \neq span(S')$$

and vice versa.

2.5 Linear Independence

In the previous section it was stated that a set of vectors S spans a given vector space V if every vector in V is expressible as a linear combination of the vectors in S. In general, it is possible that there may be more than one way to express a vector in V as a linear combination of vectors in a spanning set. This section will focus on the conditions under which each vector in V is expressible as a unique linear combination of the spanning vectors. Spanning sets with this property play a fundamental role in the study of vector spaces.

Definitions If $S = \{v_1, v_2, \dots, v_r\}$ is a nonempty set of vectors, then the vector equation

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_r\mathbf{v_r} = \mathbf{0}$$

has at least one solution, namely

$$k_1 = 0, k_2 = 0, \dots, k_r = 0$$

If this is the only solution, then S is called a **linearly independent** set. If there are other solutions, then S is called a **linearly dependent** set.

Examples

1. If $\mathbf{v_1} = (2,-1,0,3), v_2 = (1,2,5,-1)$ and $v_3 = (7,-1,5,8)$, then the set of vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent, since $3\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3} = \mathbf{0}$.

2. The polynomials

$$\mathbf{p_1} = 1 - x, \mathbf{p_2} = 5 + 3x - 2x^2, \ and\ \mathbf{p_3} = 1 + 3x - x^2$$

form a linearly dependent set in P_2 since $3\mathbf{p_1} - \mathbf{p_2} + 2\mathbf{p_3} = \mathbf{0}$

3. Consider the vectors $\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)$ in $\Re^3$. In terms of components the vector equation

$$k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$$

becomes

$$k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0)$$

or equivalently,

$$(k_1, k_2, k_3) = (0,0,0)$$

Thus the set $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is linearly independent. A similar argument can be used to extend S to a linear independent set in $\Re^n$. Try and work out what the linear independent set is in your head.

The following two theorems follow quite simply from the Definition of linear independence and linear dependence.

Theorem 2.6 *A set S with two or more vectors is:*

(a) Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

(b) Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.

Example

1. Recall that the vectors

$$\mathbf{v_1} = (2,-1,0,3), \mathbf{v_2} = (1,2,5,-1) \ \ and\ \ \mathbf{v_3} = (7,-1,5,8)$$

were linear dependent because

$$3\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3} = \mathbf{0}$$

It is obvious from the equation that

$$\mathbf{v_1} = \frac{-1}{3}\mathbf{v_2} + \frac{1}{3}\mathbf{v_3}, \quad \mathbf{v_2} = -3\mathbf{v_1} + 1\mathbf{v_3}, \quad \mathbf{v_3} = 3\mathbf{v_1} + 1\mathbf{v_2}$$

Theorem 2.7 *(a) A finite set of vectors that contains the zero vector is linearly dependent.*

(b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

2.6 Operations on Vector Spaces

Definitions

- The addition of two vector spaces is defined by: $U + V = \{\mathbf{u} + \mathbf{v} | \mathbf{u} \in U, \mathbf{v} \in V\}$
- The intersection $\cap$ of two vector spaces is defined by:

$$U \cap V = \{\mathbf{w} | \mathbf{w} \in U \ and\ \mathbf{w} \in V\}$$

3 BASIS AND DIMENSION

A line is thought of as 1-Dimensional, a plane 2-Dimensional, and surrounding space as 3-Dimensional. This section will attempt to make this intuitive notion of dimension precise and extend it to general vector spaces.

3.1 Coordinate systems of General Vector Spaces

A line is thought of as 1-Dimensional because every point on that line can be specified by 1 coordinate. In the same way a plane is thought of as 2 Dimensional because every point on that plane can be specified by 2 coordinates and so on. What defines this coordinate system? The most common form of defining a coordinate system is the use of coordinate axes. In the case of the plane the x and y axes are used most frequently. But there is also a way of specifying the coordinate system with vectors. This can be done by replacing each axis with a vector of length one that points in the positive direction of the axis. In the case of the $x-y$ plane the x and y-axes are replaced by the well known unit vectors $\mathbf{i}$ and $\mathbf{j}$ respectively. Let O be the origin of the system and P be any point in the plane. The point P can be specified by the vector $\overline{OP}$. Every vector, $\overline{OP}$ can be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$:

$$\overline{OP} = a\mathbf{i} + b\mathbf{j}$$

The coordinates of P, corresponding to this coordinate system, are (a, b).

Informally stated, vectors such as $\mathbf{i}$ and $\mathbf{j}$ that specify a coordinate system are called "basis vectors" for that system. Although in the preceding discussion our basis vectors were chosen to be of unit length and mutually perpendicular this is not essential. As long as linear combinations of the vectors chosen are capable of specifying all points in the plane. In our example this only requires that the two vectors are not collinear. Different basis vectors however do change the coordinates of a point, as the following example demonstrates.

Example Let $S = \{\mathbf{i}, \mathbf{j}\}$, $U = \{\mathbf{i}, 2\mathbf{j}\}$ and $V = \{\mathbf{i}+\mathbf{j}, \mathbf{j}\}$. Let the sets S,U and V be three sets of basis vectors. Let P be the point $\mathbf{i}+2\mathbf{j}$. The coordinates of P relative to each set of basis vectors is:

$$S \to (1, 2)$$

$$U \to (1, 1)$$

$$T \to (1, 1)$$

The following definition makes the preceding ideas more precise and enables the extension of a coordinate system to general vector spaces.

Definition

- If V is any vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a set of vectors in V, then S is called a **basis** for V if the following two conditions hold:
  (a) S is linearly independent
  (b) S spans V

A basis is the vector space generalization of a coordinate system in 2-space and 3-space. The following theorem will aid in understanding how this is so.

Theorem 3.1 *If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$ in exactly one way.*

Proof: Since S spans V, it follows from the definition of a spanning set that every vector in V is expressible as a linear combination of the vectors in S. To see that there is only one way to express a vector as a linear combination of the vectors in S, suppose that some vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

and also as

$$\mathbf{v} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_n\mathbf{v_n}$$

Subtracting the second equation from the first gives

$$\mathbf{0} = (c_1 - k_1)\mathbf{v_1} + (c_2 - k_2)\mathbf{v_2} + \cdots + (c_n - k_n)\mathbf{v_n}$$

Since the right side of this equation is a linear combination of vectors in S, the linear independence of S implies that

$$(c_1 - k_1) = 0, (c_2 - k_2) = 0, \ldots, (c_n - k_n)$$

That is

$$c_1 = k_1, c_2 = k_2, \ldots, c_n = k_n$$

Thus the two expressions for $\mathbf{v}$ are the same.

Definitions

- If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for a vector space V, and

  $$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

  is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates** of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in $\Re^n$ constructed from these coordinates is called the **coordinate vector of** v **relative to** S; it is denoted by

  $$[\mathbf{v}]_S = (c_1, c_2, \ldots, c_n)$$

- If $\mathbf{v} = [\mathbf{v}]_S$ then S is called the **standard basis**.

Remark It should be noted that coordinate vectors depend not only on the basis S but also on the order in which the basis vectors are written; a change in the order of the basis vectors results in a corresponding change of order for the entries in the coordinate vectors.

Examples

1. In example 3 of Section 2.5 it was shown that if

   $$\mathbf{i} = (1, 0, 0), \ \ \mathbf{j} = (0, 1, 0) \ \ \ and \ \ \ \mathbf{k} = (0, 0, 1)$$

   then $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a linearly independent set in $\Re^3$. This set also spans $\Re^3$ since any vector $\mathbf{v} = (a, b, c)$ can be written as

   $$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 1, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

   Thus, S is a basis for $\Re^3$. It is in fact a **standard basis** for $\Re^3$. Looking at the coefficients of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ above, it follows that the coordinates of $\mathbf{v}$ relative to the standard basis are a, b and c, so

   $$[\mathbf{v}]_S = (a, b, c)$$

   and so we have

   $$[\mathbf{v}]_S = \mathbf{v}$$

3.2 Dimension of General Vector Spaces

Definition

- A nonzero vector space V is called **finite-dimensional** if it contains a finite set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ that forms a basis. If no such set exists, V is called **infinite-dimensional**. In addition, the zero vector space is regarded as finite-dimensional.

Examples

- The vector spaces $\Re^n$ and P_n are both finite-dimensional.
- The vector space of all real valued functions defined on $(-\infty, \infty)$ is infinite-dimensional.

Theorem 3.2 *If V is finite-dimensional vector space and $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is any basis, then:*

(a) Every set with more than n vectors is linearly dependent.

(b) No set with fewer than n vectors spans V.

Proof:

(a) Let $S' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m}\}$ be any set of m vectors in V, where $m > n$. It remains to be shown that S' is linearly dependent. Since $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis for V, each $\mathbf{w_i}$ can be expressed as a linear combination of the vectors in S, say:

$$\mathbf{w_1} = a_{11}\mathbf{v_1} + a_{21}\mathbf{v_2} + \cdots + a_{n1}\mathbf{v_n}$$

$$\mathbf{w_2} = a_{12}\mathbf{v_1} + a_{22}\mathbf{v_2} + \cdots + a_{n2}\mathbf{v_n}$$

$$\vdots$$

$$\mathbf{w_m} = a_{1m}\mathbf{v_1} + a_{2m}\mathbf{v_2} + \cdots + a_{nm}\mathbf{v_n}$$

To show that S' is linearly dependent, scalars $k_1, k_2, \ldots, k_n$ must be found, not all zero, such that

$$k_1\mathbf{w_1} + k_2\mathbf{w_2} + \cdots + k_m\mathbf{w_m} = \mathbf{0}$$

combining the above 2 systems of equations gives

$$\begin{aligned}(k_1a_{11} + k_2a_{12} + \cdots + k_ma_{1m})\mathbf{v_1} \\ + (k_1a_{21} + k_2a_{22} + \cdots + k_ma_{2m})\mathbf{v_2} \\ \ddots \\ + (k_1a_{n1} + k_2a_{n2} + \cdots + k_ma_{nm})\mathbf{v_n} = \mathbf{0}\end{aligned}$$

Thus, from the linear independence of S, the problem of proving that S' is a linearly dependent set reduces to showing there are scalars $k_1, k_2, \ldots, k_m$, not all zero, that satisfy

$$a_{11}k_1 + a_{12}k_2 + \cdots + a_{1m}k_m = 0$$

$$a_{21}k_1 + a_{22}k_2 + \cdots + a_{2m}k_m = 0$$

$$\vdots$$

$$a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nm}k_m = 0$$

As the system is homogeneous and there are more unknowns than equations ($m > n$) Theorem 1.2 guarantees that there are non trivial solutions such that $k_1, k_2, \ldots, k_m$ are not all zero.

(b) Let $S' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m}\}$ be any set of m vectors in V, where $m < n$. It remains to be shown that S' does not span V. The proof is by contradiction: assume S' spans V. This leads to a contradiction of the linear dependence of the basis $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ of V.

If S' spans V, then every vector in V is a linear combination of the vectors in S'. In particular, each basis vector $\mathbf{v_i}$ is a linear combination of the vectors in S', say

$$\mathbf{v_1} = a_{11}\mathbf{w_1} + a_{21}\mathbf{w_2} + \cdots + a_{n1}\mathbf{w_m}$$

$$\mathbf{v_2} = a_{12}\mathbf{w_1} + a_{22}\mathbf{w_2} + \cdots + a_{n2}\mathbf{w_m}$$

$$\vdots$$

$$\mathbf{v_n} = a_{1n}\mathbf{w_1} + a_{2n}\mathbf{w_2} + \cdots + a_{mn}\mathbf{w_m}$$

To obtain the contradiction it will be shown that there exist scalars $k_1, k_2, \ldots, k_n$ not all zero, such that

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_n\mathbf{v_n} = \mathbf{0}$$

Observe the similarity to the above two systems compared with those given in the proof of (a). It can be seen that they are identical except that the $\mathbf{w}$'s and the $\mathbf{v}$'s and the m's and n's have been interchanged. Thus the above system in the same way again reduces to the problem of finding $k_1, k_2, \ldots, k_n$ not all zero, that satisfy

$$a_{11}k_1 + a_{12}k_2 + \cdots + a_{1m}k_n = 0$$

$$a_{21}k_1 + a_{22}k_2 + \cdots + a_{2m}k_n = 0$$

$$\vdots$$

$$a_{m1}k_1 + a_{m2}k_2 + \cdots + a_{mn}k_n = 0$$

As the system is homogeneous and there are more unknowns than equations ($n > m$) Theorem 1.2 guarantees that there are non trivial solutions such that $k_1, k_2, \ldots, k_m$ are not all zero. And hence the contradiction.

The last theorem essentially states this. Let S be a set with n vectors which forms a basis for the vector space V. Let S' be another set of vectors in V consisting of m vectors. If m is greater than n, S' cannot form a basis for V as the vectors in S' cannot be linearly independent. If m is less than n, S' cannot form a basis for V because it does not span V. Thus, Theorem 3.2 leads directly into one of the most important theorems in linear algebra.

Theorem 3.3 *All bases for a finite-dimensional vector space have the same number of vectors.*

And thus the concept of dimension is almost complete. Alls that is needed is a definition.

Definition

- The **dimension** of a finite-dimensional vector space V, denoted by $dim(V)$, is defined to be the number of vectors in a basis for V. In addition, the zero vector space has dimension zero.

Examples

1. The dimension of of some common vector spaces is given below:

$$dim(\Re^n) = n$$
$$dim(P_n) = n + 1$$
$$dim(M_{mn}) = mn$$

2. Determine a basis for and the dimension of the solution space of the homogeneous system:

$$\begin{array}{rrrrrl} 2x_1+ & 2x_2- & x_3 & & + \; x_5 = & 0 \\ -x_1- & x_2+ & 2x_3- & 3x_4+ & x_5 = & 0 \\ x_1+ & x_2- & 2x_3 & & - \; x_5 = & 0 \\ & & x_3+ & x_4+ & x_5 = & 0 \end{array}$$

It was shown in section 1.5 that the general solution is

$$x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$$

Therefore the solution vectors can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

which shows that the vectors

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad and \quad \mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

span the solution space. Since they are also linearly independent (clearly), $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis, and the solution space has a dimension of 2.

3.3 Related Theorems

The remaining part of this section states theorems which illustrate the subtle interrelationships among the concepts of spanning, linear independence, basis and dimension. These theorems form in many ways the building blocks of many further theorems in linear algebra.

Theorem 3.4 Plus/Minus Theorem. *Let S be a nonempty set of vectors in a vector space V.*

(a) If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of the span(S), then the set $S \cup \mathbf{v}$ that results by inserting $\mathbf{v}$ is still linearly independent.

(b) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space: that is,

$$span(S) = span(S - \mathbf{v})$$

A proof will not be included, but the theorem can be visualised in $\Re^3$ as follows.

(a) Consider two linearly independent vectors in $\Re^3$. These two vectors span a plane. If you add a third vector to them that is not in the plane, then the three vectors are still linearly independent and they span the entire domain of $\Re^3$.

(b) Consider three non-collinear vectors in a plane that form a set S. The set S spans the plane. If any one of the vectors is removed from S to give S' it is clear that S' still spans the plane. That is $span(S) = span(S')$.

Theorem 3.5 *If V is an n-dimensional vector space and if S is a set in V with exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.*

Proof: Assume that S has exactly n vectors and spans V. To prove that S is a basis it must be shown that S is a linearly independent set. But if this is not so, then some vector $\mathbf{v}$ in S is a linear combination of the remaining vectors. If this vector is removed from S, then it follows from the Theorem 3.4(b) that the remaining set of n-1 vectors still spans V. But this is impossible, since it follows from Theorem 3.2(b), that no set with fewer than n vectors can span an n-dimensional vector space. Thus, S is linearly independent.

Assume S has exactly n vectors and is a linearly independent set. To prove that S is a basis it must be shown that S spans V. But if this is not so, then there is some vector $\mathbf{v}$ in V that is not in span(S). If this vector is inserted in S, then it follows from the Theorem 3.4(a) that this set of n+1 vectors is still linearly independent. But this is impossible because it follows from Theorem 3.2(a) that no set with more than n vectors in an n-dimensional vector space can be linearly independent. Thus S spans V.

Examples

- $\mathbf{v_1} = (-3, 8)$ and $\mathbf{v_2} = (1, 1)$ form a basis for $\Re^2$ because $\Re^2$ has dimension two and v_1 and v_2 are linearly independent.

Theorem 3.6 *Let S be a finite set of vectors in a finite-dimensional vector space V.*

(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Proof:

(a) The proof is constructive and is called the **left to right algorithm**.

Let $\mathbf{v_{c_1}}$ be the 1^{st} nonzero vector in the set S. Choose the next vector in the list which is not a linear combination of $\mathbf{v_{c_1}}$ and call it $\mathbf{v_{c_2}}$. Find the next vector in the list which is not a linear combination of $\mathbf{v_{c_1}}$ and $\mathbf{v_{c_2}}$ and call it $\mathbf{v_{c_3}}$. Continue in such a way until the number of vectors chosen equals $dim(V)$.

(b) This proof is also constructive.

Let V be a vector space. Begin with $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$ which form a linearly independent family in V. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be a basis for V. Now it is necessary and important that $r < n$. To extend the basis, simply apply the left to right algorithm to the set (note that this set spans V because it contains a basis within it)

$$\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$$

This will select a basis for V that commences with $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$

Theorem 3.7 *If W is a subspace of a finite-dimensional vector space V, then $dim(W) \leq dim(V)$; moreover, if $dim(W) = dim(V)$, then $W = V$*

Proof: Let $S = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_r}\}$ be a basis for W. Either S is also a basis for V or it is not. If it is, then $dim(W) = dim(V) = m$. If it is not, then by the previous theorem, vectors can be added to the linearly independent set S to make it into a basis for V, so $dim(W) < dim(V)$. Thus, $dim(W) \leq dim(V)$ in all cases. If $dim(W) = dim(V)$, then S is a set of m linearly independent vectors in the m-dimensional vector space V; hence by Theorem 3.5, S is a basis for V. Therefore $W = V$.

3.4 Change of Basis

The Basis of a vector space is the set of vectors that specify the coordinate system. A vector space may have an infinite number of bases but each basis contains the same number of vectors. The number of vectors in the basis is called the dimension of the Vector Space. The coordinate vector or coordinate matrix of a point changes with any change in the basis used. If the basis for a vector space is changed from some old bases β to some new bases γ, how is the old coordinate vector $[\mathbf{v}]_\beta$ of a vector $\mathbf{v}$ related to the new coordinate vector $[\mathbf{v}]_\gamma$? The following theorem answers that question.

Theorem 3.8 *If the basis for a vector space is changed from some old basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ to some new basis $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, then the old coordinate vector $[\mathbf{w}]_\beta$ is related to the new coordinate vector $[\mathbf{w}]_\gamma$ of the same vector $\mathbf{w}$ by the equation*

$$[\mathbf{w}]_\gamma = P[\mathbf{w}]_\beta$$

where the columns of P are the coordinate vectors of the old basis vectors relative to the new basis; that is, the column vectors of P are

$$[\mathbf{u_1}]_\gamma, [\mathbf{u_2}]_\gamma, \ldots, [\mathbf{u_n}]_\gamma$$

P is called the **change of basis matrix** *or the* **change of coordinate matrix**.

Proof: Let V be a vector space with a basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ and a new basis $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$. Let $\mathbf{w} \in V$. Therefore $\mathbf{w}$ can be expressed as:

$$\mathbf{w} = a_1\mathbf{u_1} + a_2\mathbf{u_2} + \cdots + a_n\mathbf{u_n}$$

Thus we have

$$[\mathbf{w}]_\beta = (a_1, a_2, \ldots, a_n)$$

As γ is also a basis of V the elements of β can be expressed as follows

$$\begin{aligned}
\mathbf{u_1} &= p_{11}\mathbf{v_1} + p_{21}\mathbf{v_2} + \cdots + p_{n1}\mathbf{v_n} \\
\mathbf{u_2} &= p_{12}\mathbf{v_1} + p_{22}\mathbf{v_2} + \cdots + p_{n2}\mathbf{v_n} \\
&\vdots \\
\mathbf{u_n} &= p_{1n}\mathbf{v_1} + p_{2n}\mathbf{v_2} + \cdots + p_{nn}\mathbf{v_n}
\end{aligned}$$

Combining this system of equations with the above expression for $\mathbf{w}$ gives

$$\begin{aligned}
\mathbf{w} = &(p_{11}a_1 + p_{12}a_2 + \cdots + p_{1n}a_n)\mathbf{v_1} \\
&+(p_{21}a_1 + p_{22}a_2 + \cdots + p_{2n}a_n)\mathbf{v_2}+ \\
&\vdots \\
&+(p_{n1}a_1 + p_{n2}a_2 + \cdots + p_{nn}a_n)\mathbf{v_n}+
\end{aligned}$$

and thus it can be seen that

$$[\mathbf{w}]_\gamma = \begin{bmatrix} p_{11}a_1 + p_{12}a_2 + \cdots + p_{1n}a_n \\ p_{21}a_1 + p_{22}a_2 + \cdots + p_{2n}a_n \\ \vdots \\ p_{n1}a_1 + p_{n2}a_2 + \cdots + p_{nn}a_n \end{bmatrix}$$

which can be written as

$$[\mathbf{w}]_\gamma = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

¿From which it can be seen

$$[\mathbf{w}]_\gamma = P[\mathbf{w}]_\beta$$

where P's columns are

$$[\mathbf{u_1}]_\gamma, [\mathbf{u_2}]_\gamma, \ldots, [\mathbf{u_n}]_\gamma$$

Example

1. Consider the bases $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$ and $\beta = \{\mathbf{u_1}, \mathbf{u_2}\}$ for $\Re^2$, where

$$\mathbf{v_1} = (1,0); \;\; \mathbf{v_2} = (0,1); \;\; \mathbf{u_1} = (1,1); \;\; \mathbf{u_2} = (2,1)$$

   (a) Find the transition matrix form β to γ. First the coordinate vectors of the old basis vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ must be found relative to the new basis β. By inspection:

$$\mathbf{u_1} = \mathbf{v_1} + \mathbf{v_2}$$

$$\mathbf{u_2} = 2\mathbf{v_1} + \mathbf{v_2}$$

   so that

$$[\mathbf{u_1}]_\gamma = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad and \quad [\mathbf{u_2}]_\gamma = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   Thus the transition matrix from β to γ

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

   (b) Use the transition matrix to find $[\mathbf{v}]_\gamma$ if

$$[\mathbf{v}]_\beta = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

   It is known from the Change of Basis theorem that

$$[\mathbf{v}]_\gamma = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

   As a check it should be possible to recover the vector $\mathbf{v}$ either from $[\mathbf{v}]_\beta$ or $[\mathbf{v}]_\gamma$. It is left for the student to show that $-3\mathbf{u_1} + 5\mathbf{u_2} = 7\mathbf{v_1} + 2\mathbf{v_2} = (7,2)$.

4 ORTHONORMAL BASES

In many problems involving vector spaces, the problem solver is free to choose any basis for the vector space that seems appropriate. In $\Re^n$ the solution of a problem is often greatly simplified by choosing a basis in which the vectors are orthogonal to one another. in this section it shall be shown how such a basis can be obtained.

Before proceeding however, it is necessary to review some basic definitions

Definitions

- Let $\mathbf{u}, \mathbf{v} \in \Re^n$. Then the operation defined by

  $$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

  where

  $$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

  maps $\Re^n \to \Re$ and is referred to as the **Euclidean inner product** or the **dot product**.

- Two vectors $\mathbf{u}, \mathbf{v} \in \Re^n$ are said to be **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

- If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors and both $\mathbf{u}$ and $\mathbf{v}$ have a magnitude of one, then $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthonormal**.

- A set of vectors in $\Re^n$, on which the dot product is defined, is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has a magnitude of one is called an **orthonormal set**.

Now that the groundwork has been laid the following theorem can be stated. The proof of this result is extremely important, since it prides an algorithm, or method, for converting an arbitrary basis into an orthonormal basis.

Theorem 4.1 *Every subspace of the vector space $\Re^n$ has an orthonormal basis.*

Proof: Let V be any nonzero subspace of $\Re^n$, and let $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}\}$ be any basis for V. It suffices to show that V has an orthogonal basis, since the vectors in the orthogonal basis can be normalized to produce an orthonormal basis for V. The following sequence of steps will produce an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ for V.

Step 1 Let $\mathbf{v_1} = \mathbf{u_1}$.

Step 2 Obtain a vector $\mathbf{v_2}$ that is orthogonal to $\mathbf{v_1}$ by computing the component of $\mathbf{u_2}$ that is orthogonal to the space W_1 spanned by $\mathbf{v_1}$. This can be done using the formula:

$$\mathbf{v_2} = \mathbf{u_2} - \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1}$$

Of course, if $\mathbf{v_2} = \mathbf{0}$, then $\mathbf{v_2}$ is not a basis vector. But this cannot happen, since it would then follow from the preceding formula for $\mathbf{v_2}$ that

$$\mathbf{u_2} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \right) \mathbf{u_1}$$

which says that $\mathbf{u_2}$ is a multiple of $\mathbf{u_1}$, contradicting the linear independence of the basis $S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$.

Step 3 To construct a vector $\mathbf{v_3}$ that is orthogonal to both $\mathbf{v_1}$ and $\mathbf{v_2}$, compute the component of $\mathbf{u_3}$ orthogonal to the space W_2 spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$ using the formula:

$$\mathbf{v_3} = \mathbf{u_3} - \left(\frac{\mathbf{u_3}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} - \left(\frac{\mathbf{u_3}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\right)\mathbf{v_2}$$

As in step 2, the linear independence of $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ ensures that $\mathbf{v_3} \neq \mathbf{0}$. The remaining details are left as an exercise.

Step 4 To determine a vector $\mathbf{v_4}$ that is orthogonal to $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$, compute the component of $\mathbf{u_4}$ orthogonal to the space W_3 spanned by $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ using the formula

$$\mathbf{v_4} = \mathbf{u_4} - \left(\frac{\mathbf{u_4}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} - \left(\frac{\mathbf{u_4}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\right)\mathbf{v_2} - \left(\frac{\mathbf{u_4}\cdot\mathbf{v_3}}{\mathbf{v_3}\cdot\mathbf{v_3}}\right)\mathbf{v_3}$$

Continuing in this way, an orthogonal set of vectors, $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$, will be obtained after m steps. Since V is an m-dimensional vector space and every orthogonal set is linearly independent, the set $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is an orthogonal basis for V.

This preceding step-by-step construction for converting an arbitrary basis into an orthogonal basis is called the **Gramm-Schmidt process**.

Examples: *THE GRAMM-SCHMIDT PROCESS*

1. Consider the vector space $\Re^3$ with the Euclidean inner product. Apply the Gramm-Schmidt process to transform the basis vectors $\mathbf{u_1} = (1,1,1), \mathbf{u_2} = (0,1,1), \mathbf{u_3} = (0,0,1)$ into an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$; then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}\}$.

   Step 1

   $$\mathbf{v_1} = \mathbf{u_1} = (1,1,1)$$

   Step 2

   $$\begin{aligned} \mathbf{v_2} &= \mathbf{u_2} - \left(\frac{\mathbf{u_2}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} \\ &= (0,1,1) - \frac{2}{3}(1,1,1) = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right) \end{aligned}$$

   Step 3

   $$\begin{aligned} \mathbf{v_3} &= \mathbf{u_3} - \left(\frac{\mathbf{u_3}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} - \left(\frac{\mathbf{u_3}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\right)\mathbf{v_2} \\ &= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= \left(0, -\frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

   Thus,

   $$\mathbf{v_1} = (1,1,1), \quad \mathbf{v_2} = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v_3} = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

   form an orthogonal basis for $\Re^3$. The norms of these vectors are

   $$\|\mathbf{v_1}\| = \sqrt{3}, \quad \|\mathbf{v_2}\| = \frac{\sqrt{6}}{3}, \quad \|\mathbf{v_3}\| = \frac{1}{\sqrt{2}}$$

so an orthonormal basis for $\Re^3$ is

$$\mathbf{q_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad \mathbf{q_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\mathbf{q_3} = \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

The Gramm-Schmidt process with subsequent normalization not only converts an arbitrary basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ into an orthonormal basis $\{\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}\}$, but it does it in such a way that for $k \geq 2$ the following relationships hold:

- $\{\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_k}\}$ is an orthonormal basis for the space spanned by $\{\mathbf{u_1}, \ldots, \mathbf{u_k}\}$.
- $\mathbf{q_k}$ is orthogonal to $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{k-1}}\}$.

The proofs are omitted but these facts should become evident after some thoughtful examination of the proof of Theorem 4.1.

5 NULL, COLUMN AND ROW SPACE

This section applies all of the previous groundwork laid in the course to study three important vector spaces associated with a matrix. This leads to a deeper understanding of the relationship between the solutions of a linear system and the properties of the coefficient matrix.

5.1 Basic Definitions

Definitions

- For an $m \times n$ matrix

  $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

  the vectors

  $$\begin{aligned} \mathbf{r_1} &= [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ \mathbf{r_2} &= [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ \vdots & \qquad\qquad \vdots \\ \mathbf{r_m} &= [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{aligned}$$

  in $\Re^n$ formed from the rows of A are called the **row vectors** of A.

- The vectors

  $$\mathbf{c_1} = \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots \quad \mathbf{c_n} = \begin{bmatrix} a_{1n} \\ a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

  in $\Re^m$ formed from the columns of A are called the **column vectors** of A.

- The subspace of $\Re^n$ spanned by the row vectors of A is called the **row space** of A. The row space of A is denoted by $R(A)$.

  $$R(A) = span\{\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}\}$$

- The subspace of $\Re^m$ spanned by the column vectors of A is called the **column space** of A. The column space of A is denoted by $C(A)$.

  $$C(A) = span\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{r_n}\}$$

- The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of $\Re^m$, is called the **null space** of A. The null space of A is denoted by $N(A)$.

  $$N(A) = \{\mathbf{x} \in \Re^n | A\mathbf{x} = \mathbf{0}\}$$

Proof:

(a) The row space and column space are subspaces. Let A be an $n \times m$ matrix. Consider the row space firstly. Every row vector is an element of $\Re^n$. As $\Re^n$ is a vector space it is closed under addition and scalar multiplication. Therefore it can be stated that

$$span\{\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}\} \subset \Re^n$$

Let $\mathbf{u}$ and $\mathbf{v} \in R(A)$. Then clearly

$$\mathbf{u} + \mathbf{v} \in span\{\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_n}\}$$
$$k\mathbf{u} \in span\{\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_n}\}$$

where k is any scalar. Therefore by Theorem 2.3 the row space is a subspace of $\Re^n$. The same argument can be used of the column space to show it is a subspace of $\Re^m$.

(b) The Null space of A is a subspace of $\Re^n$. Clearly $N(A) \subset \Re^n$. Let $\mathbf{u}$ and $\mathbf{v} \in N(A)$. Then

$$A\mathbf{u} = \mathbf{0} \quad and \quad A\mathbf{v} = \mathbf{0}$$

therefore

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$
$$\mathbf{u} + \mathbf{v} \in N(A)$$

also if k is any scalar

$$A(k\mathbf{u}) = kA\mathbf{u} = k\mathbf{0} = \mathbf{0}$$
$$k\mathbf{u} \in N(A)$$

Hence the null space is closed under addition and scalar multiplication and is therefore, by Theorem 2.3, a subspace of $\Re^n$.

Example Let A be an 2×3 matrix as shown below

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 9 & 2 \end{bmatrix}$$

Then the row vectors and column vectors are

$$\mathbf{r_1} = [1, 3, 2] \quad and \quad \mathbf{r_2} = [4, 9, 2]$$
$$\mathbf{c_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{c_2} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \mathbf{c_3} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

And quite obviously we have

$$R(A) = span\{\mathbf{r_1}, \mathbf{r_2}\}$$
$$C(A) = span\{\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}\}$$

Finding the Null space of A amounts to finding the solution space of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$. Forming the augmented matrix $[A \mid \mathbf{0}]$ gives

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 9 & 2 & 0 \end{bmatrix}$$

This reduces to the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

It can be seen that x_3 is the independent variable. Let $x_3 = t$. Then the general solution becomes

$$x_2 = -2t \quad and \quad x_1 = 4t$$

Thus it can be said that

$$\mathbf{x} = t \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Therefore $N(A)$ is spanned by the vector $(4, 2, 1)^T$.

The next question to ask is: How is it possible to find a basis for these three subspaces? Finding a basis for the N(A) is simple. It just amounts to finding the solution space of the homogeneous system. Several examples of this procedure have been given previously. The next section looks at an algorithm used for determining bases for R(A) and C(A).

5.2 Finding Bases for R(A) and C(A)

Before stating the Algorithm for forming a basis for the row space and the column space of a matrix it is necessary to state a few theorems regarding the effect of elementary row operations on these subspaces.

Elementary row operations were first developed as an aid in solving linear systems. It was discovered that performing an elementary row operation on an augmented matrix does not change the solution set of the corresponding linear system. It follows that applying an elementary row operation to a matrix A does not change the solution set of the corresponding linear system $A\mathbf{x} = \mathbf{0}$, or, stated another way, it does not change the null space of A. Thus, the following theorem states:

Theorem 5.1 *Elementary row operations do not change the null space of a matrix*

In fact we have already used this theorem in the previous example. The following theorem is a companion to Theorem 5.1.

Theorem 5.2 *Elementary row operations do not change the row space of a matrix.*

Proof: Suppose that the row vectors of a matrix A are $\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}$ and let B be obtained from A by performing an elementary row operation. It shall be shown that every vector in the row space of B is also in the row space of A, and conversely that every vector in the row space of A is in the row space of B. It can thus be concluded that A and B have the same row space.

Consider the possibilities: If the row operation is a row interchange, then B and A have the same row vectors and consequently have the same row space. If the row operation is multiplication of a row by a nonzero scalar or the addition of a multiple of one row to another, then the row vectors $\mathbf{r'_1}, \mathbf{r'_2}, \ldots, \mathbf{r'_m}$ of B are linear combinations of $\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m}$; thus, they lie in the row space of A. Since a vector space is closed under addition and scalar multiplication, all linear combinations of $\mathbf{r'_1}, \mathbf{r'_2}, \ldots, \mathbf{r'_m}$ will also lie in the row space of A. Therefore, each vector in the row space of B is in the row space of A.

Since B is obtained from A by performing a row operation, A can be obtained from B by performing the inverse operation. Thus, the argument above, when reversed, shows that the row space of A is contained in the row space of B.

In light of the previous two theorems, one might be forgiven for thinking that elementary row operations have no effect on the column space of a matrix. It is in fact, definitely not the case. Elementary row operations on a matrix do change the column space. For example, consider the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

The second column is a scalar multiple of the first, so the column space of A consists of all scalar multiples of the first column vector. However, if we add -2 times the first row of A to the second row, we obtain

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

Here again the second column is a scalar multiple of the first, so the column space of B consists of all scalar multiples of the first column vector. This is not the same as the column space of A.

Although elementary row operations can change the column space of a matrix, it shall be shown that whatever relationships of linear independence or linear dependence exist between the column vectors prior to a row operation also exist between the column vectors of the resulting matrix. To show this, let B be the matrix that results by performing an elementary row operation on an $m \times n$ matrix A. By theorem 4.1 the two homogeneous linear systems

$$A\mathbf{x} = \mathbf{0} \quad \textit{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set. Thus, the first system has a nontrivial solution if and only if the same is true of the second. To make things even more plane, let the column vectors of A and B, respectively be

$$\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n} \quad \textit{and} \quad \mathbf{c'_1}, \mathbf{c'_2}, \ldots, \mathbf{c'_n}$$

then the two systems can be rewritten as

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n} = \mathbf{0} \tag{1}$$

$$x_1\mathbf{c'_1} + x_2\mathbf{c'_2} + \cdots + x_n\mathbf{c'_n} = \mathbf{0} \tag{2}$$

Thus, (1) has a nontrivial solution for $x_1, x_2, \ldots, x_n$ if and only if the same is true of (2). This implies that the column vectors of A are linearly independent if and only if the same is true of B. Although the proof is omitted, this conclusion can also be applied to any subset of the column vectors. Thus, we have the following result.

Theorem 5.3 *If A and B are row equivalent matrices, then:*

(a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

(b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

The three previous theorems lead to the following theorem which makes it possible to find bases for the row and column spaces of a matrix in row-echelon form by inspection.

Theorem 5.4 *If a matrix R is in row-echelon form, then the row vectors with the leading 1's (that is, the non-zero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.*

This result is virtually self-evident from examples. Hence the proof is omitted and replaced with several examples. The proof simply involves an analysis of the positions of the 0's and the 1's of R.

Examples

1. The matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row-echelon form. From Theorem 5.4 the vectors

$$\begin{aligned} \mathbf{r_1} &= [1 \quad -2 \quad 5 \quad 0 \quad 3] \\ \mathbf{r_2} &= [0 \quad 1 \quad 3 \quad 0 \quad 0] \\ \mathbf{r_3} &= [0 \quad 0 \quad 0 \quad 1 \quad 0] \end{aligned}$$

form a basis for the row space of R, and the vectors

$$\mathbf{c_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c_2} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

2. Find the bases for the row and column space of

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Since the elementary row operations do not change the row space of a matrix, a basis for the row space of A can be found by finding a basis for the row space of any row-echelon form of A. Reducing A to row-echelon form one obtains

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By theorem 4.4 the nonzero row vectors of R form a basis for the row space of R, and hence form a basis for the row space of A. These basis vectors are

$$\begin{aligned} \mathbf{r_{1_R}} &= [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4] \\ \mathbf{r_{2_R}} &= [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6] \\ \mathbf{r_{3_R}} &= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5] \end{aligned}$$

Keeping in mind that A and R may have different column spaces, it is not possible to find a basis for the column space of A *directly* from the column vectors of R. However, it follows from Theorem 5.4 that if it is possible to find a set of column vectors of R that form a basis for the column space of R, then the *corresponding* column vectors of A will form a basis for the column space of A. The first, third, and fifth columns of R contain the leading 1's of the row vectors, so

$$\mathbf{c_{1_R}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c_{3_R}} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c_{5_R}} = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R; thus the corresponding column vectors of A, namely,

$$\mathbf{c_{1_A}} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c_{3_A}} = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad \mathbf{c_{5_A}} = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

3. Find a basis for the space spanned by the vectors

$$\mathbf{v_1} = (1, -2, 0, 0, 3), \quad \mathbf{v_2} = (2, -5, -3, -2, 6), \quad \mathbf{v_3} = (0, 5, 15, 10, 0)$$

$$\mathbf{v_4} = (2, 6, 18, 8, 6)$$

The space spanned by these vectors is exactly the same as the row space of the following matrix. As the space spanned is identical, if a basis is found for the row space of the matrix it will also be a row space for the above set of vectors.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

The matrix is reduced to row-echelon form, which gives

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors in this matrix are

$$\mathbf{w_1} = (1, -2, 0, 0, 3), \quad \mathbf{w_2} = (0, 1, 3, 2, 0), \quad \mathbf{w_3} = (0, 0, 1, 1, 0)$$

These vectors form a basis for the row space and consequently form a basis for the subspace of Re^5 spanned by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ and $\mathbf{v_4}$.

Note that in the previous two examples that the basis for the column space of A consisted of column vectors of A while the row space of A consisted of row vectors of row-echelon form of A. The following example demonstrates how to find a basis for the space spanned by a set of vectors that consists entirely of the vectors in the original set. Naturally instead of using the row space, one uses the column space to obtain a basis.

4. (a) Find a subset of the vectors

$$\mathbf{v_1} = (1, -2, 0, 3), \quad \mathbf{v_2} = (2, -5, -3, 6)$$

$$\mathbf{v_3} = (0, 1, 3, 0), \quad \mathbf{v_4} = (2, -1, 4, -7), \quad \mathbf{v_5} = (5, -8, 1, 2)$$

that forms a basis for the space spanned by these vectors.

Begin by constructing a matrix that has $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_5}$ as its column vectors. Let this matrix be A.

$$\textit{Let } A = [\mathbf{v_1} \mid \mathbf{v_2} \mid \mathbf{v_3} \mid \mathbf{v_4} \mid \mathbf{v_5}]$$

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 0 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

The problem is easily solved by finding a basis for the column space of this matrix. For reasons that will become evident in part (b) the matrix is reduced to *reduced* row-echelon form. Denoting the column vectors of the resulting matrix by $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ and $\mathbf{w_5}$ yields

$$\textit{Let } B = rref(A) = [\mathbf{w_1} \mid \mathbf{w_2} \mid \mathbf{w_3} \mid \mathbf{w_4} \mid \mathbf{w_5}]$$

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1's occur in columns 1, 2 and 4, so that by Theorem 5.4

$$\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_4}$$

is a basis for the column space of B and consequently

$$\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}$$

form a basis for the column space of the A and hence the original system

(b) Express the vectors not in the basis as a linear combination of the basis vectors.

Here is the reason for the reduced row-echelon form in part (a). It is now so much easier to see the linear dependence of $\mathbf{w_3}$ and $\mathbf{w_5}$ on the other vectors when the matrix is reduced to reduced row-echelon form. By inspection, the linear combinations are

$$\mathbf{w_3} = 2\mathbf{w_1} - \mathbf{w_2}$$
$$\mathbf{w_5} = \mathbf{w_1} + \mathbf{w_2} + \mathbf{w_4}$$

We call these the **dependency equations**. The corresponding relationships in the original matrix are:

$$\mathbf{v_3} = 2\mathbf{v_1} - \mathbf{v_2}$$
$$\mathbf{v_5} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_4}$$

5.2.1 Extending a Linearly Independent set to form a Basis

It was stated in Theorem 3.6(b) that if V is a vector space of dimension n and if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ form an linearly independent set in V such that $r < n$ then there exists vectors $\mathbf{v_{r+1}}, \mathbf{v_{r+2}}, \ldots, \mathbf{v_n}$ such that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ form a basis for V. The next example demonstrates how to extend an linearly independent family to a basis for V.

Example

1. Take $V = \Re^4$, and also let $\mathbf{v_1}$ and $\mathbf{v_2}$ be

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 2 \\ 6 \\ -8 \\ 100 \end{bmatrix}$$

It is clear that $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. As specified in the proof of Theorem 3.6(b), the set $\mathbf{v_1}, \mathbf{v_2}, \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ is formed, where $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$ is the standard basis for V. Let A be the matrix

$$A = [\mathbf{v_1} \mid \mathbf{v_2} \mid \mathbf{u_1} \mid \mathbf{u_2} \mid \mathbf{u_3} \mid \mathbf{u_4}]$$

If the basis for the column space of A is found, the linearly independent set will have been extended to form a basis for V. Note that the set spans all of $\Re^4$ and will therefore have a basis of $\Re^4$ by Theorem 3.6(a). Finding the rref of A.

$$B = rref(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Therefore it can be seen that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{u_1}, \mathbf{u_2}$ form a basis for $\Re^4$ and thus the linearly independent set has been extended to form a basis.

5.3 Rank and Nullity

Theorem 5.5 *If A is any matrix, then the row space and column space of A have the same dimension*

Proof: Let R be the reduced row-echelon form of A. It follows from Theorem 5.2 that

$$\dim(\text{row space of } A) = \dim(\text{row space of } R)$$

and it follows from Theorem 5.3 that

$$\dim(\text{column space of } A) = \dim(\text{column space of } R)$$

Thus, the proof would be complete if it could be shown that the row space and column space of the matrix R have the same dimension. The dimension of the row space of R is the number of nonzero rows and the dimension of the column space of R is the number of columns that contain leading 1's (Theorem 5.4). However, the nonzero rows are precisely the rows in which the leading 1's occur, so the number of leading 1's and the number of nonzero rows is the same. This shows that the row space and column space of R have the same dimension.

The dimension of the null, column and row space are extremely important numbers. Because of this, there is special notation and terminology associated with them.

Definitions

- The common dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by $\text{rank}(A)$.
- The dimension of the null space of A is called the **nullity** of A and is denoted by $\text{nullity}(A)$.

The following theorem establishes an important relationship between the rank and nullity of a matrix.

Theorem 5.6 Dimension Theorem for Matrices. *If A is a matrix with n columns, then*

$$\textit{rank}(A) + \textit{nullity}(A) = n$$

Proof: Since A has n columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has n unknowns (variables). These fall into two categories: the leading variables and the free variables. Thus,

$$\begin{bmatrix} \textit{number of leading} \\ \textit{variables} \end{bmatrix} + \begin{bmatrix} \textit{number of free} \\ \textit{variables} \end{bmatrix} = n$$

But the number of leading variables is the same as the number of leading 1's in the reduced row-echelon form of A, and this is the rank of A. Thus,

$$\textit{rank}(A) + \begin{bmatrix} \textit{number of free} \\ \textit{variables} \end{bmatrix} = n$$

The number of free variables is equal to nullity of A. This is so because the nullity of A is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, which is the same as the number of parameters in the general solution, which is the same as the number of free variables. Thus,

$$\text{rank}(A) + \text{nullity}(A) = n$$

5.4 Related Theorems

The following theorem provides conditions under which a linear system of m equations in n unknowns is guaranteed to be consistent.

Theorem 5.7 (The Consistency Theorem). *If $A\mathbf{x} = \mathbf{0}$ is a linear system of m equations in n unknowns, then the following are equivalent.*

(a) $A\mathbf{x} = \mathbf{b}$ *is consistent.*

(b) $\mathbf{b}$ *is in the column space of A.*

(c) The coefficient matrix A and the augmented matrix $[A \mid \mathbf{b}]$ have the same rank.

Proof: It suffices to prove the two equivalencies (a)$\Leftrightarrow$(b) and (b)$\Leftrightarrow$(c), since it will then follow from logic that (a)$\Leftrightarrow$(c).

(a)⇔(b) Consider the system $A\mathbf{x} = \mathbf{b}$. Let A be an $m \times n$ matrix. Therefore in general

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \textit{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$ denote the column vectors of A, then the product $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is expressible as a linear combination of these column vectors with coefficients from $\mathbf{x}$; that is,

$$A\mathbf{x} = x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n}$$

Thus, a linear system, $A\mathbf{x} = \mathbf{b}$, of m equations in n unknowns can be written as

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n} = \mathbf{b}$$

from which it can be concluded that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is expressible as a linear combination of the column vectors of A or, equivalently, if and only if $\mathbf{b}$ is in the column space of A.

(b)⇒(c) It shall be shown that if $\mathbf{b}$ is in the column space of A, then the column spaces of A and $[A \mid \mathbf{b}]$ are actually the same, from which it follows that the two matrices have the same rank.

By definition, the column space of a matrix is the space spanned by its column vectors, so the column spaces of A and $[A \mid \mathbf{b}]$ can be expressed as

$$\text{span}\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}\} \text{ and } \text{span}\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}, \mathbf{b}\}$$

respectively. If $\mathbf{b}$ is in the column space of A, then each vector in the set $\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}, \mathbf{b}\}$ is a linear combination of the vectors in $\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}\}$ and conversely. Thus, from Theorem 2.5 the column spaces of A and $[A \mid \mathbf{b}]$ are the same.

(c)⇒(b) Assume that A and $[A \mid \mathbf{b}]$ have the same rank r. By Theorem 3.6(a), there is some subset of the column vectors of A that forms a basis for the column space of A. Suppose that those column vectors are

$$\mathbf{c'_1}, \mathbf{c'_2}, \ldots, \mathbf{c'_r}$$

These r basis vectors also belong to the r-dimensional column-space of $[A|\mathbf{b}]$ by Theorem 3.6(a). This means that $\mathbf{b}$ is expressible as a linear combination of $\mathbf{c'_1}, \mathbf{c'_2}, \ldots, \mathbf{c'_r}$, and consequently $\mathbf{b}$ lies in the column space of A.

Examples This theorem is best demonstrated by the following example. It helps to think of the rank of a matrix as the number of nonzero rows in its reduced row-echelon form. The augmented matrix for the system

$$\begin{array}{rrrrr} x_1- & 2x_2- & 3x_3+ & 2x_4 = & -4 \\ -3x_1+ & 7x_2- & x_3+ & x_4 = & -3 \\ 2x_1- & 5x_2+ & 4x_3- & 3x_4 = & 7 \\ -3x_1+ & 6x_2+ & 9x_3- & 6x_4 = & -1 \end{array}$$

is

$$\begin{bmatrix} 1 & -2 & -3 & 2 & -4 \\ -3 & 7 & -1 & 1 & -3 \\ 2 & -5 & 4 & -3 & 7 \\ -3 & 6 & 9 & -6 & -1 \end{bmatrix}$$

which has the following reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & -23 & 16 & 0 \\ 0 & 1 & -10 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Because of the third row the system is inconsistent. However, it is also because of this row that the reduced row-echelon form of the augmented matrix has fewer zero rows than the reduced row-echelon form of the coefficient matrix. This forces the coefficient matrix and the augmented matrix for the system to have different ranks.

The Consistency Theorem was concerned with the conditions under which a linear system $A\mathbf{x} = \mathbf{b}$ is consistent for a *specific* vector $\mathbf{b}$. The following theorem is concerned with conditions under which a linear system is consistent for *all possible* choices of $\mathbf{b}$.

Theorem 5.8 *If $A\mathbf{x} = \mathbf{b}$ is a linear system of m equations in n unknowns, then the following are equivalent.*

(a) $A\mathbf{x} = \mathbf{b}$ is consistent for every $m \times 1$ matrix $\mathbf{b}$.

(b) The column vectors of A span $\Re^m$.

(c) $rank(A) = m$.

Proof: It suffices to prove the two equivalencies (a)⇔(b) and (a)⇔(c), since it will then follow from logic that (b)⇔(c).

(a)⇔(b) It is known that $A\mathbf{x} = \mathbf{b}$ can be expressed as

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n} = \mathbf{b}$$

from which it can be concluded that $A\mathbf{x} = \mathbf{b}$ is consistent for every $m \times 1$ matrix $\mathbf{b}$ if and only if every such $\mathbf{b}$ is expressible as a linear combination of the column vectors $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$, or equivalently, if and only if these column vectors span $\Re^m$.

(a)⇒(c) ¿From the assumption that $A\mathbf{x} = \mathbf{b}$ is consistent for every $m \times 1$ matrix $\mathbf{b}$, and from parts (a) and (b) of Theorem 5.7, it follows that every vector $\mathbf{b}$ in $\Re^m$ lies in the column space of A; that is, the column space of A is all of $\Re^m$. Thus rank(A) = dim$(\Re^m) = m$.

(c)⇒(a) ¿From the assumption that rank$(A) = m$, it follows that the column space of A is a subspace of $\Re^m$ of dimension m, and hence must be all of $\Re^m$ by Theorem 3.7. It now follows from parts (a) and (b) of Theorem 5.7 that $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\Re^m$, since every such $\mathbf{b}$ is in the column space of A.

Example

1. A linear system with more equations than unknowns is called an **overdetermined linear system.** If $A\mathbf{x} = \mathbf{b}$ is an overdetermined linear system of m equations in n unknowns (so that $m > n$), then the column vectors of A cannot span $\Re^m$. This is due to the fact, that if the columns were going to span $\Re^m$ there would have to be at least m of them (Theorem 3.2). However there are only n of them, and therefore the column space does not span $\Re^m$.

Theorem 5.9 *If A is an $m \times n$ matrix, then the following are equivalent.*

(a) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(b) The column vectors of A are linearly independent.

(c) $A\mathbf{x} = \mathbf{b}$ has at most one solution (none or one) for every $m \times 1$ matrix $\mathbf{b}$.

Proof: It suffices to prove the two equivalencies (a)⇔(b) and (a)⇔(c), since it will then follow from logic that (b)⇔(c).

(a)⇔(b) If $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$ are the column vectors of A, then the linear system $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \cdots + x_n\mathbf{c_n} = \mathbf{0} \tag{3}$$

If $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$ are linearly independent, then this equation is satisfied only by $x_1 = x_2 = \cdots = x_n = 0$, which means that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then (3) is satisfied only by $x_1 = x_2 = \cdots = x_n = 0$, which means that $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$ are linearly independent.

(a)⇒(c) Assume that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Either $A\mathbf{x} = \mathbf{b}$ is consistent or it is not. If it is not consistent, then there are no solutions of $A\mathbf{x} = \mathbf{b}$, and the first section of the proof is complete. If $A\mathbf{x} = \mathbf{b}$ is consistent, let $\mathbf{x_0}$ be any solution. Using the fact that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, it is concluded that the general solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x_0} + \mathbf{0} = \mathbf{x_0}$. Thus, the only solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x_0}$.

(c)⇒(a) Assume that $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $m \times 1$ matrix $\mathbf{b}$. Then, in particular, $A\mathbf{x} = \mathbf{0}$ has at most one solution. Thus, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

The last few theorems have dealt with $m \times n$ matrices. This next theorem deals with square matrices exclusively and their invertibility.

Theorem 5.10 *If A is an $n \times n$ matrix, then the following are equivalent.*

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(c) The column vectors of A are linearly independent

(d) The row vectors of A are linearly independent

(e) The column vectors of A span $\Re^n$.

(f) The row vectors of A span $\Re^n$.

(g) The column vectors of A form a basis for $\Re^n$.

(h) The row vectors of A form a basis for $\Re^n$.

(i) A has rank n

(j) A has nullity 0.

Proof: To complete the proof it shall be shown that (a) ⇔ (b) and that (c) through (h) are equivalent to (b) by proving the sequence of implications $(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (b)$

(a)⇔(b)

Assume A is invertible and let $\mathbf{x_0}$ be any solution of $A\mathbf{x} = \mathbf{0}$; thus, $A\mathbf{x_0} = \mathbf{0}$. Multiplying both sides of this equation by the matrix A^{-1} gives $A^{-1}(A\mathbf{x_0}) = A^{-1}\mathbf{0}$, or $(A^{-1}A)\mathbf{x_0} = \mathbf{0}$, or $I\mathbf{x_0} = \mathbf{0}$, or $\mathbf{x_0} = \mathbf{0}$. Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b)⇒(c)

If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then by Theorem 5.9 (previous) the column vectors of A are linearly independent.

(c)⇒(d)⇒(e)⇒(f) ⇒(g)⇒(h)

This follows from Theorem 3.5 and the fact that $\Re^n$ is an n-dimensional vector space.

(h)⇒(i)

If the n row vectors of A form a basis for $\Re^n$, then the row space of A is n-dimensional and A has rank n.

(i)⇒(j)

This follows from the Dimension Theorem 5.6.

(j)⇒(b)

If A has nullity 0, then the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension 0, which means that it contains only the zero vector. Hence, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

6 LINEAR TRANSFORMATIONS

Definitions

- If $T : V \to W$ is a function that maps a vector space V into a vector space W, then T is called a **linear transformation** from V to W if for all vectors $\mathbf{u}$ and $\mathbf{v}$ in V and all scalars c

  (a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

  (b) $T(c\mathbf{u}) = cT(\mathbf{u})$

- In the special case where $V = W$, the linear transformation $T : V \to V$ is called a **linear operator** on V.

- Let A be an $m \times n$ matrix and let $T : \Re^n \to \Re^m$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \Re^n$. Then as a matter of notational convention it is said that T is the linear transformation T_A.

Examples

1. T_A is a linear transformation. Let A be an $m \times n$ matrix and let $T : \Re^n \to \Re^m$ be the linear transformation defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \Re^n$. Let $\mathbf{u}$ and $\mathbf{v} \in \Re^n$, then

$$T(\lambda\mathbf{u} + \mu\mathbf{v}) = A(\lambda\mathbf{u} + \mu\mathbf{v})$$

$$= \lambda A\mathbf{u} + \mu A\mathbf{v}$$

$$= \lambda T_A(\mathbf{u}) + \mu T_A(\mathbf{v})$$

   and thus T_A is a linear transformation.

2. If I is the $n \times n$ identity matrix, then for every vector $\mathbf{x}$ in $\Re^n$

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

   so multiplication by I maps every vector in $\Re^n$ into itself. $T_I(\mathbf{x})$ is called the **identity operator** on $\Re^n$.

3. Let A, B and X be $n \times n$ matrices. Then $Y = AX - XB$ is also $n \times n$.

   Let $V = M_{n\times n}(\Re)$ be the vector space of all $n \times n$ matrices. Then $Y = AX - XB$ defines a transformation $T : V \to V$. The transformation is linear for:

$$\begin{aligned} T(\lambda X_1 + \mu X_2) &= A(\lambda X_1 + \mu X_2) - (\lambda X_1 + \mu X_2)B \\ &= \lambda AX_1 + \mu AX_2 - \lambda X_1 b - \mu X_2 B \\ &= \lambda(AX_1 - X_1B) + \mu(AX_2 - X_2B) \\ &= \lambda T(X_1) + \mu T(X_2) \end{aligned}$$

Theorem 6.1 *If $T : \Re^n \to \Re^m$ is a linear transformation, then there exists an $m \times n$ matrix A such that $T = T_A$.*

Proof: Let $\mathbf{x} \in \Re^n$ and let $T : \Re^n \to \Re^m$. Also let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ be the standard basis for $\Re^n$. Then it can be stated.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}$$

Then

$$\begin{aligned} T(\mathbf{x}) &= x_1 T(\mathbf{e_1}) + x_2 T(\mathbf{e_2}) + \cdots + x_n T(\mathbf{e_n}) \\ &= [T(\mathbf{e_1}) \mid T(\mathbf{e_2}) \mid \cdots \mid T(\mathbf{e_n})] \\ &= A\mathbf{x} \end{aligned}$$

where A is the matrix whose columns are the transformation of $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$. Therefore $T = T_A$.

Example

1. Find the 2×2 matrix A such that $T = T_A$ has the property that

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It is necessary to find $T(\mathbf{e_1})$ and $T(\mathbf{e_2})$, because as outlined in the preceding proof the matrix $A = [T(\mathbf{e_1}) \mid T(\mathbf{e_2})]$. It can be seen that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{e_1} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

And it can also be seen that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{e_2} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So due to the linearity of the linear transformation it can be stated

$$T(\mathbf{e_1}) = -T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and it can also be stated that

$$T(\mathbf{e_2}) = 2T\begin{bmatrix} 1 \\ 1 \end{bmatrix} + -T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Thus it is clear that

$$A = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

6.1 Geometric Transformations in $\Re^2$

This section consists of various different transformations of the form T_A that have a geometrical interpretation. Such transformations form the building blocks for understanding linear transformations.

Examples of Geometric Transformations

- Operators on $\Re^2$ and $\Re^3$ that map each vector into its symmetric image about some line or plane are called **reflection operators**. Such operators are of the form T_A and are thus linear. There are three main reflections in $\Re^2$. These are summarised below. Considering the transformation from the coordinates (x, y) to (w_1, w_2) the properties of the operator are as follows.

1. **Reflection about the y-axis**: The equations for this transformation are

$$\begin{aligned} w_1 &= -x \\ w_2 &= y \end{aligned}$$

The standard matrix for the transformation is clearly

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

To demonstrate the reflection, consider the below example.

$$\text{Let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{therefore } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

2. **Reflection about the x-axis**: The equations for this transformation are

$$\begin{aligned} w_1 &= x \\ w_2 &= -y \end{aligned}$$

The standard matrix for the transformation is clearly

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

To demonstrate the reflection, consider the below example.

$$\text{Let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{therefore } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

3. **Reflection about the line $y = x$**: The equations for this transformation are

$$\begin{aligned} w_1 &= y \\ w_2 &= x \end{aligned}$$

The standard matrix for the transformation is clearly

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To demonstrate the reflection, consider the below example.

$$\text{Let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{therefore } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Operators on $\Re^2$ and $\Re^3$ that map each vector into its orthogonal projection on a line or plane through the origin are called **orthogonal projection operators**. Such operators are of the form T_A and are thus linear. There are two main projections in $\Re^2$. These are summarised below. Considering the transformation from the coordinates (x, y) to (w_1, w_2) the properties of the operator are as follows.

  1. **Orthogonal projection onto the x-axis**: The equations for this transformation are

  $$\begin{aligned} w_1 &= x \\ w_2 &= 0 \end{aligned}$$

  The standard matrix for the transformation is clearly

  $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  To demonstrate the projection, consider the below example.

  $$\text{Let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

  $$\text{therefore } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  2. **Orthogonal projection on the y-axis**: The equations for this transformation are

  $$\begin{aligned} w_1 &= 0 \\ w_2 &= y \end{aligned}$$

  The standard matrix for the transformation is clearly

  $$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

  To demonstrate the projection, consider the below example.

  $$\text{Let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

  $$\text{therefore } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- An operator that rotates each vector in $\Re^2$, through a fixed angle θ is called a **rotation operator** on $\Re^2$. Such operators are of the form T_A and are thus linear. There is only one rotation in $\Re^2$, due to the generality of the formula. This rotation is summarised below. Considering the transformation from the coordinates (x, y) to (w_1, w_2) the properties of the operator are as follows.

  1. **Rotation through an angle θ**: The equations for this transformation are

  $$\begin{aligned} w_1 &= x\cos\theta - y\sin\theta \\ w_2 &= x\sin\theta + y\cos\theta \end{aligned}$$

  The standard matrix for the transformation is clearly

  $$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To demonstrate the projection, consider the below example.

$$\text{Let } \theta = 30^\circ \text{ and let } \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{therefore } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

- If k is a nonnegative scalar, then the operator $T(\mathbf{x}) = k\mathbf{x}$ on $\Re^2$ and $\Re^3$ is called a **contraction with factor** k if $0 \leq k \leq 1$, and a **dilation with factor** k if $k \geq 1$. Such operators are of the form T_A and are thus linear. The contraction and the dilation operators are summarised below. Considering the transformation from the coordinates (x, y) to (w_1, w_2) the properties of the operator are as follows.

  1. **Contraction with factor** k **on** $\Re^2$**,** $(0 \leq k \leq 1)$: The equations for this transformation are.

     $$\begin{aligned} w_1 &= kx \\ w_2 &= ky \end{aligned}$$

     The standard matrix for the transformation is clearly

     $$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

     To demonstrate the contraction, consider the below example.

     $$\text{Let } k = \frac{1}{2} \text{ and let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

     $$\text{therefore } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

  2. **Dilation with factor** k **on** $\Re^2$**,** $(k \geq 1)$: The equations for this transformation are

     $$\begin{aligned} w_1 &= kx \\ w_2 &= ky \end{aligned}$$

     The standard matrix for the transformation is clearly

     $$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

     To demonstrate the dilation, consider the below example.

     $$\text{Let } k = 2 \text{ and let } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

     $$\text{therefore } T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

6.2 Basic Properties of Linear Transformations

Theorem 6.2 *If $T : V \to W$ is a linear transformation, then:*

(a) $T(\mathbf{0}) = \mathbf{0}$

(b) $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v}$ *in* V.

(c) $T(\mathbf{v} - \mathbf{w}) = T(\mathbf{v}) - T(\mathbf{w})$ *for all* $\mathbf{v}$ *and* $\mathbf{w}$ *in* V.

Proof:

(a) Let $\mathbf{v}$ be any vector in V. Since $0\mathbf{v} = \mathbf{0}$, it can be seen that

$$T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}$$

(b) Also,

$$T(-\mathbf{v}) = T((-1)\mathbf{v}) = (-1)T(\mathbf{v}) = -T(\mathbf{v})$$

(c) Finally, $\mathbf{v} - \mathbf{w} = \mathbf{v} + -1\mathbf{w}$; thus,

$$\begin{aligned} T(\mathbf{v} - \mathbf{w}) &= T(\mathbf{v} + -1\mathbf{w}) \\ &= T(\mathbf{v})(-1)T(\mathbf{w}) \\ &= T(\mathbf{v}) - T(\mathbf{w}) \end{aligned}$$

In words, part (a) of the Theorem 6.2 states that a linear transformation maps $\mathbf{0}$ into $\mathbf{0}$. This property is useful for identifying transformations that are not linear.

6.3 Product of Linear Transformations

Definition

- If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, the **composite of T_2 with T_1** denoted by $T_2 \circ T_1$, is the function defined by the formula

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$$

where $\mathbf{u}$ is a vector in U.

Remark: Observe that this definition requires the domain of T_2 (which is V) to contain the range of T_1; this is essential for the formula $T_2(T_1(\mathbf{u}))$ to make sense.

The next result shows that the composition of two linear transformations is itself a linear transformation.

Theorem 6.3 *If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.*

Proof: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in U and s and t are scalars, then it follows from the definition of a composite transformation and from the linearity of T_1 and T_2 that

$$\begin{aligned} T_2 \circ T_1(s\mathbf{u} + t\mathbf{v}) &= T_2(T_1(s\mathbf{u} + t\mathbf{v})) \\ &= T_2(sT_1(\mathbf{u}) + tT_1(\mathbf{v})) \\ &= sT_2(T_1(\mathbf{u})) + tT_2(T_1(\mathbf{v})) \\ &= sT_2 \circ T_1(\mathbf{u}) + tT_2 \circ T_1(\mathbf{v}) \end{aligned}$$

And thus the proof is complete

Examples

1. Let A be an $m \times n$ matrix, and B be an $n \times p$ matrix, then AB is an $m \times p$ matrix. Also $T_A : \Re^n \to \Re^m$, and $T_B : \Re^p \to \Re^n$ are both linear transformations. Then

$$\begin{aligned} T_A \circ T_B &= T_A(T_B(\mathbf{x})) \\ &= AB\mathbf{x} \\ &= (AB)\mathbf{x} \\ &= T_{AB}(\mathbf{x}) \end{aligned}$$

where $\mathbf{x} \in \Re^p$. And therefore $T_A \circ T_B = T_{AB} : \Re^p \to \Re^m$.

2. If V has a basis $\beta = \{\mathbf{v_1}, \mathbf{v_2}\}$ and $T : V \to V$ is a linear transformation given by

$$\begin{aligned} T(\mathbf{v_1}) &= 2\mathbf{v_1} + 3\mathbf{v_2} \\ T(\mathbf{v_2}) &= -7\mathbf{v_1} + 8\mathbf{v_2} \end{aligned}$$

To find $T \circ T(-\mathbf{v_1} + 3\mathbf{v_2})$ takes two steps as shown below.

$$\begin{aligned} T(-\mathbf{v_1} + 3\mathbf{v_2}) &= -T(\mathbf{v_1}) + 3T(\mathbf{v_2}) \\ &= -2\mathbf{v_1} - 3\mathbf{v_2} + 3(-7\mathbf{v_1} + 8\mathbf{v_2}) \\ &= -23\mathbf{v_1} + 21\mathbf{v_2} \end{aligned}$$

Hence

$$\begin{aligned} T \circ T(-\mathbf{v_1} + 3\mathbf{v_2}) &= T(-23\mathbf{v_1} + 21\mathbf{v_2}) \\ &= -23T(\mathbf{v_1}) + 21T(\mathbf{v_2}) \\ &= -23(2\mathbf{v_1} + 3\mathbf{v_2}) + 21(-7\mathbf{v_1} + 8\mathbf{v_2}) \\ &= -193\mathbf{v_1} + 99\mathbf{v_2} \end{aligned}$$

6.4 Kernel and Image

Recall that if A is an $m \times n$ matrix, then the null-space of A consists of all vectors $\mathbf{x}$ in $\Re^n$ such that $A\mathbf{x} = \mathbf{0}$, and by Theorem 5.7 the column space of A consists of all vectors $\mathbf{b}$ in $\Re^m$ for which there is at least one vector $\mathbf{x}$ in $\Re^n$ such that $A\mathbf{x} = \mathbf{b}$. From the viewpoint of matrix transformations, the null-space of A consists of all vectors in $\Re^n$ that multiplication by A maps in to $\mathbf{0}$, and the column space of A consists of all vectors in $\Re^m$ that are images of at least one vector in $\Re^n$ under multiplication by A. The following definition extends these ideas to general linear transformations.
Definitions

- If $T : V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the **kernel** of T. It is denoted by ker(T). In mathematical notation:

$$ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

- If $T : V \to W$ is a linear transformation, then the set of all vectors in W that are images under T of at least one vector in V is called the **Image** (or range in some texts) of T; it is denoted by Im(T). In mathematical notation:

$$Im(T) = \{\mathbf{w} \in W \mid \mathbf{v} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in U\}$$

Examples

1. If $T_A : \Re^n \to \Re^m$ is multiplication by the $m \times n$ matrix A, then from the discussion preceding the definition above, the kernel of T_A is the null-space of A, and the image of T_A is the column space of A.

2. Let $I : V \to V$ be the identity operator. Since $I\mathbf{v} = \mathbf{v}$ for all vectors in V, every vector in V is the image of some vector (namely, itself); thus, $Im(I) = V$. Since the *only* vector that I maps into $\mathbf{0}$ is $\mathbf{0}$, it follows that $ker(I) = \{\mathbf{0}\}$.

3. Let $T : \Re^3 \to \Re^3$ be the orthogonal projection on the $x - y$ plane. The kernel of T is the set of points that T maps into $\mathbf{0} = (0, 0, 0)$; these are the points on the z-axis. Since T maps every point in $\Re^3$ into the $x - y$ plane, the image of T must be some subset of this plane. But every point $(x_0, y_0, 0)$ in the $x - y$ plane is the image under T of some point; in fact, it is the image of all points on the vertical line that passes through $(x_0, y_0, 0)$. Thus $Im(T)$ is the entire $x - y$ plane.

4. Let $T : \Re^2 \to \Re^2$ be the linear operator that rotates each vector in the $x - y$ plane through the angle θ. Since *every* vector in the $x - y$ plane can be obtain by rotating some vector through the angle θ, one obtains $Im(T) = \Re^2$. Moreover, the only vector that rotates into $\mathbf{0}$ is $\mathbf{0}$, so $ker(T) = \{\mathbf{0}\}$.

5. Find the kernel of the linear transformation $T : \Re^3 \to \Re^3$ given by
$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$
It is thus necessary to find the vectors $[x, y, z]^T$ such that
$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
That is to say
$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
Therefore it is clear that $x = y = 0$ and z is arbitrary. Therefore the set of vectors
$$\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$
with z arbitrary is equal to the $\ker(T)$. It is clear that the standard basis vector $\mathbf{e_3}$ forms a basis for the $\ker(T)$.

In all of the preceding examples, $ker(T)$ and $Im(T)$ turned out to be *subspaces*. This is no accident as the following theorem points out.

Theorem 6.4 *If $T : V \to W$ is a linear transformation, then:*

(a) The kernel of T is a subspace of V.

(b) The range of T is a subspace of W.

Proof:

(a) To show that $ker(T)$ is a subspace, it must be shown that it contains at least one vector and is closed under addition and scalar multiplication. By part (a) of Theorem 6.2, the vector $\mathbf{0}$ is in $ker(T)$, so this set contains at least one vector. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be vectors in $ker(T)$, and let k be any scalar. Then

$$T(\mathbf{v_1} + \mathbf{v_2}) = T(\mathbf{v_1}) + T(\mathbf{v_2}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so that $\mathbf{v_1}$ and $\mathbf{v_2}$ is in $ker(T)$. Also,

$$T(k\mathbf{v_1}) = kT(\mathbf{v_1}) = k\mathbf{0} = \mathbf{0}$$

so that $k\mathbf{v_1}$ is in $ker(T)$.

(b) Since $T(\mathbf{0}) = \mathbf{0}$, there is at least one vector in $Im(T)$. Let $\mathbf{w_1}$ and $\mathbf{w_2}$ be vectors in the range of T, and let k be any scalar. To prove this part it must be shown that $\mathbf{w_1} + \mathbf{w_2}$ and $k\mathbf{w_1}$ are in the range of T; that is, vectors $\mathbf{a}$ and $\mathbf{b}$ must be found in V such that $T(\mathbf{a}) = \mathbf{w_1} + \mathbf{w_2}$ and $T(\mathbf{b}) = k\mathbf{w_1}$.

Since $\mathbf{w_1}$ and $\mathbf{w_2}$ are in the range of T, there are vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ in V such that $T(\mathbf{a_1}) = \mathbf{w_1}$ and $T(\mathbf{a_2}) = \mathbf{w_2}$. Let $\mathbf{a} = \mathbf{a_1} + \mathbf{a_2}$ and $\mathbf{b} = k\mathbf{a_1}$. Then

$$T(\mathbf{a}) = T(\mathbf{a_1} + \mathbf{a_2}) = T(\mathbf{a_1}) + T(\mathbf{a_2}) = \mathbf{w_1} + \mathbf{w_2}$$

and

$$T(\mathbf{b}) = T(k\mathbf{a_1}) = kT(\mathbf{a_1}) = k\mathbf{w_1}$$

which completes the proof.

Theorem 6.5 *If $T : U \to V$ is a linear transformation and $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ forms a basis for U, then $Im(T) = span(T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n}))$*

This theorem is best demonstrated by a simple example.

Example

Let A be $m \times n$ and let $T = T_A$. Then $T_A : \Re^n \to \Re^m$. Let $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$ be the standard basis for $\Re^n$. Then by the previous theorem it can be stated

$$\begin{aligned} Im(T_A) &= span(T_A(\mathbf{e_1}), T_A(\mathbf{e_2}), \ldots, T_A(\mathbf{e_n})) \\ &= span(A\mathbf{e_1}, A\mathbf{e_2}, \ldots, A\mathbf{e_n}) \\ &= span(col_1(A), col_2(A), \ldots, col_n(A)) \\ &= C(A) \end{aligned}$$

6.5 Rank and Nullity

In Section 4 of this course the rank of a matrix was defined to be the dimension of its column (or row) space and the nullity to be the dimension of its null-space. The following definition extends these definitions to general linear transformations.

Definitions If $T : V \to W$ is a linear transformation,

- then the dimension of the image of T is called the **rank of** T and is denoted by rank(T),
- and the dimension of the kernel is called the **nullity of** T and is denoted by nullity(T).

Examples

- Let U be a vector space of dimension n, with basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$, and let $T : U \to V$ be a linear transformation defined by

$$T(\mathbf{u_1}) = \mathbf{u_2}, T(\mathbf{u_2}) = \mathbf{u_3}, \cdots, T(\mathbf{u_n}) = \mathbf{u_n} \text{ and } T(\mathbf{u_n}) = \mathbf{0}$$

  Find bases for Ker(T) and Im(T) and determine rank(T) and nullity(T).

  Using the result from Theorem 6.5 it can be stated

$$\begin{aligned} Im(T) &= span(T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n})) \\ &= span(\mathbf{u_2}, \mathbf{u_3}, \ldots, \mathbf{u_n}) \end{aligned}$$

  So $\mathbf{u_2}, \mathbf{u_3}, \ldots, \mathbf{u_n}$ form a basis for Im(T) and hence the rank(T) $= n - 1$.

  It is quite clear that $\mathbf{u_n} \in ker(T)$, as $T(\mathbf{u_n}) = \mathbf{0}$. So $span(\mathbf{u_n}) \subset ker(T)$. Conversely suppose that $\mathbf{u} \in ker(T)$, it is shown that $\mathbf{u} \in span(\mathbf{u})$. Then $T(\mathbf{u}) = \mathbf{0}$. As $\mathbf{u}$ is an element of U it can be expressed as a linear combination of the basis vectors of U, as shown below

$$\mathbf{u} = x_1\mathbf{u_1} + x_2\mathbf{u_2} + \cdots + x_n\mathbf{u_n}$$

  Therefore it can be stated

$$\mathbf{0} = T(\mathbf{u}) = x_1T(\mathbf{u_1}) + x_2T(\mathbf{u_2}) + \cdots + x_nT(\mathbf{u_n})$$

$$= x_1\mathbf{u_2} + x_2\mathbf{u_3} + \cdots + x_{n-1}\mathbf{u_n} + \mathbf{0}$$

  Due to the linear independence of $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$, $x_1, x_2, \ldots, x_{n-1} = 0$. Therefore

$$\mathbf{u} = x_n\mathbf{u_n}$$

  Hence ker(T) = span($\mathbf{u_n}$), and therefore the nullity(T) is equal to one.

If A is an $m \times n$ matrix and $T_A : \Re^n \to \Re^m$ is multiplication by A, then it is known from a previous example that the kernel of T_A is the null-space of A and the range of T_A is the column space of A. Thus, the following relationship exists between the rank and nullity of a matrix and the rank and nullity of the corresponding matrix transformation.

Theorem 6.6 *If A is an $m \times n$ matrix and $T_A : \Re^n \to \Re^m$ is multiplication by A, then:*

(a) nullity(T_A) = nullity(A)

(b) rank(T_A) = rank(A)

Recall from Theorem 5.6 that if A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

The following theorem extends this result to general linear transformations.

Theorem 6.7 *If $T : V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then*

$$rank(T) + nullity(T) = dim(V) = n$$

Proof: The proof is divided up into two cases.

Case 1 Let U be the zero vector space. Then due to theorem 5.2 it is known that $T(\mathbf{0}) = \mathbf{0}$. Therefore it can be stated that

$$\text{Im}(T) = \{\mathbf{0}\} \text{ and } \ker(T) = \{\mathbf{0}\}$$

therefore

$$\text{rank}(T)+\text{nullity}(T) = 0 + 0 = 0 = \dim(U)$$

Case 2 Let U be an n-dimensional vector space with the basis $\{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}\}$. Then the proof can be divided up into three parts.

(a) Consider the case where $\ker(T) = \{\mathbf{0}\}$. Let $\mathbf{u} \in \ker(T)$. As $\mathbf{u} \in U$ it can be expressed as

$$\mathbf{u} = x_1\mathbf{u_1} + x_2\mathbf{u_2} + \cdots + x_n\mathbf{u_n} \tag{4}$$

As $\mathbf{u} \in \ker(T)$ it can be stated that

$$\mathbf{0} = T(\mathbf{u}) = x_1T(\mathbf{u_1}) + x_2T(\mathbf{u_2}) + \cdots + x_nT(\mathbf{u_n}) \tag{5}$$

Due to the fact that $\ker(T) = \{\mathbf{0}\}$, $\mathbf{u} = \mathbf{0}$. Due to the linear independence of $\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}$ it follows from equation (4) that $x_1, x_2, \dots, x_n = 0$. It then also follows from equation (5) that $T(\mathbf{u_1}), T(\mathbf{u_2}), \dots, T(\mathbf{u_n})$ are linearly independent. It is known from Theorem 6.4 that $\text{Im}(T) = \text{span}(T(\mathbf{u_1}), T(\mathbf{u_2}), \dots, T(\mathbf{u_n}))$. As $T(\mathbf{u_1}), T(\mathbf{u_2}), \dots, T(\mathbf{u_n})$ are linearly independent they form a basis for $\text{Im}(T)$. It can therefore be stated that

$$\text{rank}(T)+\text{nullity}(T) = n + 0 = n = \dim(U)$$

(b) Consider the case where $\ker(T) = U$. Theorem 6.4 states: $\text{Im}(T) = \text{span}(T(\mathbf{u_1}), T(\mathbf{u_2}), \dots, T(\mathbf{u_n}))$. However $\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n} \in \ker(T)$. Therefore $T(\mathbf{u_1})$, $T(\mathbf{u_2}), \dots, T(\mathbf{u_n}) = \mathbf{0}$. So it can be stated that $\text{Im}(T) = \text{span}(\mathbf{0}) = \{\mathbf{0}\}$. Therefore

$$\text{rank}(T)+\text{nullity}(T) = 0 + n = n = \dim(U)$$

(c) Consider the case where $1 \leq \text{nullity}(T) < n$. Assume that the $\text{nullity}(T) = r$, and let $\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_r}$ be a basis for the kernel. Since $\{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_r}\}$ form a linearly independent set, Theorem 3.6(b) states that there are $n - r$ vectors, $\mathbf{u_{r+1}}, \mathbf{u_{r+2}}, \dots, \mathbf{u_n}$, such that $\{\mathbf{u_1}, \dots, \mathbf{u_r}, \mathbf{u_{r+1}}, \dots, \mathbf{u_n}\}$ is a basis for U. To complete the proof it shall be shown that the $n - r$ vectors in the set $S = \{T(\mathbf{u_{r+1}}), \dots, T(\mathbf{u_n})\}$ form a basis for the image of T. It then follows that

$$\text{rank}(T)+\text{nullity}(T) = n - r + r = n = \dim(U)$$

First it shall be shown that S spans the range of T. If $\mathbf{b}$ is any vector in the range of T, the $\mathbf{b} = T(\mathbf{u})$ for some vector $\mathbf{u}$ in U. Since $\{\mathbf{u_1}, \dots, \mathbf{u_r}, \mathbf{u_{r+1}}, \dots, \mathbf{u_n}\}$ is a basis for U, the vector $\mathbf{u}$ can be written in the form

$$\mathbf{u} = c_1\mathbf{u_1} + \cdots + c_r\mathbf{u_r} + c_{r+1}\mathbf{u_{r+1}} + \cdots + c_n\mathbf{u_n}$$

since $\mathbf{u_1}, \dots, \mathbf{u_r}$ lie in the kernel of T, it is clear that $T(\mathbf{u_1}), \dots, T(\mathbf{u_r}) = \mathbf{0}$, so that

$$\mathbf{b} = T(\mathbf{u}) = c_{r+1}T(\mathbf{u_{r+1}}) + \cdots + c_nT(\mathbf{u_n})$$

Thus, S spans the image of T.

Finally, it shall be shown that S is a linearly independent set and consequently forms a basis for the range of T. Suppose that some linear combination of the vectors in S is zero; that is,

$$k_{r+1}T(\mathbf{u_{r+1}}) + \cdots + k_nT(\mathbf{u_n}) = \mathbf{0} \tag{6}$$

It must be shown that $k_{r+1} = \cdots = k_n = 0$. Since T is linear, equation (6) can be rewritten as

$$T(k_{r+1}\mathbf{u_{r+1}} + \cdots + k_n\mathbf{u_n}) = \mathbf{0}$$

which says that $k_{r+1}\mathbf{u_{r+1}} + \cdots + k_n\mathbf{u_n}$ is in the kernel of T. This vector can therefore be written as a linear combination of the basis vectors $\{\mathbf{u_1}, \ldots, \mathbf{u_r}\}$, say

$$k_{r+1}\mathbf{u_{r+1}} + \cdots + k_n\mathbf{u_n} = k_1\mathbf{u_1} + \cdots + k_r\mathbf{u_r}$$

Thus,

$$k_1\mathbf{u_1} + \cdots + k_r\mathbf{u_r} - k_{r+1}\mathbf{u_{r+1}} - \cdots - k_n\mathbf{u_n} = \mathbf{0}$$

Since $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ is linearly independent, all of the k's are zero; in particular $k_{r+1} = \cdots = k_n = 0$, which completes the proof.

Examples Let $T : \Re^2 \to \Re^2$ be the linear operator that rotates each vector in the $x - y$ plane through an angle of θ. It was showed previously that $\ker(T) = \{\mathbf{0}\}$ and $\mathrm{Im}(T) = \Re^2$. Thus,

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = 2 + 0 = 2 = \dim(U)$$

which is consistent with the fact that the domain of T is two-dimensional.

6.6 Matrix of a Linear Transformation

In this section it shall be shown that if V and W are finite-dimensional vector spaces, then with a little ingenuity any linear transformation $T : V \to W$ can be regarded as a matrix transformation. The basic idea is to work with coordinate matrices of the vectors rather than with the vectors themselves.

Definition

- Suppose that V is an n-dimensional vector space and W an m-dimensional vector space. Let β and γ be bases for V and W respectively, then for each $\mathbf{x}$ in V, the coordinate vector $[\mathbf{x}]_\beta$ will be a vector in $\Re^n$, and the coordinate vector $[T(\mathbf{x})]_\gamma$ will be a vector in $\Re^m$. If there exists an $m \times n$ matrix A, such that

$$A[\mathbf{x}]_\beta = [T(\mathbf{x})]_\gamma \tag{7}$$

then A is called **the matrix of the transformation relative to bases β and γ** and it is written

$$A = [T]_\beta^\gamma$$

Theorem 6.8 *Let $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ and $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ be bases for the vector spaces V and W respectively, and let $\mathbf{x} \in V$. If $T : V \to W$ is a linear transformation then*

(a) the matrix of transformation relative to bases β and γ always exists. That is to say, there always exists a matrix $A = [T]_\beta^\gamma$ such that

$$A[\mathbf{x}]_\beta = [T(\mathbf{x})]_\gamma$$

(b) The matrix of transformation relative to basis β and γ has the form

$$[T]_\beta^\gamma = [[T(\mathbf{u_1})]_\gamma \mid [T(\mathbf{u_2})]_\gamma \mid \cdots \mid [T(\mathbf{u_n})]_\gamma]$$

Proof: Let $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ be a basis for the n-dimensional space V and let $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ be a basis for the m-dimensional space W. Then the matrix $[T]_\beta^\gamma = A$ must have the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

such that (7) holds for all vectors $\mathbf{x}$ in V. In particular, this equation must hold for the basis vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$; that is,

$$A[\mathbf{u_1}]_\beta = [T(\mathbf{u_1})]_\gamma, A[\mathbf{u_2}]_\beta = [T(\mathbf{u_2})]_\gamma, \ldots, A[\mathbf{u_n}]_\beta = [T(\mathbf{u_n})]_\gamma \tag{8}$$

But

$$[\mathbf{u_1}]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, [\mathbf{u_2}]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, [\mathbf{u_n}]_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

so

$$A[\mathbf{u_1}]_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$A[\mathbf{u_2}]_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$$

$$\vdots$$

$$A[\mathbf{u_n}]_\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Substituting these results into equation (7) yields

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = [T(\mathbf{u_1})]_\gamma, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} = [T(\mathbf{u_2})]_\gamma, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = [T(\mathbf{u_n})]_\gamma$$

which shows that the successive columns of A are coordinate vectors of

$$T(\mathbf{u_1}), T(\mathbf{u_2}), \ldots, T(\mathbf{u_n})$$

with respect to the basis γ. Thus the matrix for T with respect to the bases β and γ is

$$[T]_\beta^\gamma = [[T(\mathbf{u_1})]_\gamma \mid [T(\mathbf{u_2})]_\gamma \mid \cdots \mid [T(\mathbf{u_n})]_\gamma]$$

Thus the proof is complete.

Examples

1. Let $T_B : \Re^n \to \Re^m$ be a linear transformation defined by $T(X) = BX$ where B is an $m \times n$ matrix. Let $\beta = \{\mathbf{E_1}, \mathbf{E_2}, \ldots, \mathbf{E_n}\}$ be the standard basis for $\Re^n$ and let $\gamma = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}\}$ be the standard basis for $\Re^m$. Then it is known from the previous theorem that $[T]_\beta^\gamma$ is the following matrix
$$[T_B]_\beta^\gamma = [[T_B(\mathbf{E_1})]_\gamma \mid [T_B(\mathbf{E_2})]_\gamma \mid \cdots \mid [T_B(\mathbf{E_n})]_\gamma]$$
In general, for $1 \leq j \leq n$ it follows from the definition of the transformation that
$$T_B(\mathbf{E_j}) = B\mathbf{E_j} = col_j(B) = b_{1j}\mathbf{e_1} + b_{2j}\mathbf{e_2} + \cdots + b_{mj}\mathbf{e_m}$$
therefore
$$[T_B(\mathbf{E_j})]_\gamma = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = col_j(B)$$
and thus it is clear that
$$[T_B]_\beta^\gamma = B$$

2. Let U have the basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and let V have the basis $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$. Let T be the linear transformation defined by
$$T(\mathbf{u_1}) = 2\mathbf{v_1} + \mathbf{v_2},\ T(\mathbf{u_2}) = \mathbf{v_1} - \mathbf{v_2},\ T(\mathbf{u_3}) = 2\mathbf{v_2}$$
Then clearly
$$[T]_\beta^\gamma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

3. Let $V = M_{2\times 2}(\Re)$ and let $T : V \to V$ be the linear transformation given by $T(X) = BX - XB$ where $X \in V$ and
$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard basis for V where
$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
To find $[T]_\beta^\beta$ it is necessary to do the following calculations:
$$T(E_{11}) = BE_{11} - E_{11}B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} = 0E_{11} + -bE_{12} + cE_{21} + 0E_{22}$$
$$T(E_{12}) = BE_{12} - E_{12}B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} -c & a-d \\ 0 & c \end{bmatrix} = -cE_{11} + (a-d)E_{12} + 0E_{21} + cE_{22}$$
$$T(E_{21}) = BE_{21} - E_{21}B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} b & 0 \\ d-a & -b \end{bmatrix} = bE_{11} + 0E_{12} + (d-a)E_{21} + -bE_{22}$$

$$T(E_{22}) = BE_{22} - E_{22}B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} = 0E_{11} + bE_{12} + -cE_{21} + 0E_{22}$$

Therefore it follows that

$$[T]_\beta^\beta = \begin{bmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{bmatrix}$$

The following theorem follows directly from the definition of the matrix of a linear transformation.

Theorem 6.9 *Let $T : V \to W$ be a linear transformation, and let β and γ be bases for V and W respectively. Then if $\mathbf{v} \in V$*

$$[T(\mathbf{v})]_\gamma = [T]_\beta^\gamma [\mathbf{v}]_\beta$$

Examples

1. Let U have the basis $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and let V have the basis $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$. Let T be the linear transformation defined by

$$T(\mathbf{u_1}) = 2\mathbf{v_1} + \mathbf{v_2},\ T(\mathbf{u_2}) = \mathbf{v_1} - \mathbf{v_2},\ T(\mathbf{u_3}) = 2\mathbf{v_2}$$

Given that $\mathbf{u} = 3\mathbf{u_1} + -2\mathbf{u_2} + 7\mathbf{u_3}$

$$[\mathbf{u}]_\beta = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}, \text{ and } [T]_\beta^\gamma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

Hence

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_{beta} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 19 \end{bmatrix}$$

Hence $T(\mathbf{u}) = 4\mathbf{v_1} + 19\mathbf{v_2}$.

The following theorem gives a recipe for finding bases for ker(T) and the Im(T) where possible.

Theorem 6.10 *Let A be an $m \times n$ matrix such that $A = [T]_\beta^\gamma$, where $T : V \to W$ is a linear transformation and $\beta = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ and $\gamma = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}\}$ are bases for V and W respectively. Let $s =$ nullity(A) and $r =$ rank(A). Then suppose that*

$$\mathbf{x_j} = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{bmatrix}$$

for $1 \le j \le s$, form a basis for $N(A)$, while $col_{c_1}(A), col_{c_2}(A), \ldots, col_{c_t}(A)$ form a basis for $C(A)$ Then

1. (a) *the vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_s}$ defined by*

$$\mathbf{u_j} = x_{1j}\mathbf{v_1} + x_{2j}\mathbf{v_2} + \cdots + x_{nj}\mathbf{v_n}$$

will be a basis for the kernel of T.

(b) *the vectors* $T(\mathbf{u_{c_1}}), T(\mathbf{u_{c_2}}), \ldots, T(\mathbf{u_{c_r}})$ *form a basis for the Image of* T.

2. *If* $N(A) = \{\mathbf{0}\}$, *then* $ker(T) = \{\mathbf{0}\}$
   If $C(A) = \{\mathbf{0}\}$, *then* $Im(T) = \{\mathbf{0}\}$

3. *Thus it follows that* $rank(T) = rank(A)$ *and* $nullity(T) = nullity(A)$.

The proof of this theorem is not included. The logic behind the theorem is easily seen however in the following example.

Example

1. Let $T : V \to W$ be a linear transformation, and let $\beta = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and $\gamma = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be bases for V and W respectively. T is the linear transformation given by

$$\begin{aligned} T(\mathbf{v_1}) &= \mathbf{w_1} + \mathbf{w_2} + -\mathbf{w_3} \\ T(\mathbf{v_2}) &= 2\mathbf{w_1} + -3\mathbf{w_2} \\ T(\mathbf{v_3}) &= 3\mathbf{w_1} + -2\mathbf{w_2} + -\mathbf{w_3} \end{aligned}$$

   Let $A = [T]_\beta^\gamma$. Therefore

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & -2 \\ -1 & 0 & -1 \end{bmatrix}$$

   Let $B = rref(A)$, then

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   It is then straight forward to see that

$$\mathbf{x_1} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

   is a basis for $N(A)$, and so it follows by Theorem 6.9 that $-\mathbf{u_1} - \mathbf{u_2} + \mathbf{u_3}$ is a basis for the kernel of T.

   It can also be seen that $col_1(A), col_2(A)$ form a basis for $C(A)$. So $T(\mathbf{u_1})$ and $T(\mathbf{u_2})$ form a basis for Im(T).

6.7 Similar Matrices

The matrix of a linear operator $T : V \to V$ depends on the basis selected for V. One of the fundamental problems of linear algebra is to choose a basis for V that makes the matrix for T as simple as possible - diagonal or triangular, for example. This section is devoted to the study of this problem

To demonstrate that certain bases produce a much simpler matrix of transformation then others, consider the following example.

Example

1. Standard bases do not necessarily produce the simplest matrices for linear operators. For example, consider the linear operator $T : \Re^2 \to \Re^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$$

and the standard basis $\beta = \{\mathbf{e_1}, \mathbf{e_2}\}$ for $\Re^2$, where

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

By Theorem 6.7, the matrix for T with respect to this basis is the standard matrix for T; that is,

$$[T]_\beta^\beta = [T(\mathbf{e_1}) \mid T(\mathbf{e_2})$$

¿From the definition of the linear transformation T,

$$T(\mathbf{e_1}) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

so

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

In comparison, consider the basis $\gamma = \{\mathbf{u_1}, \mathbf{u_2}\}$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

By Theorem 6.7, the matrix for T with respect to the basis γ is

$$[T]_\gamma^\gamma = [[T(\mathbf{u_1})]_\gamma \mid [T(\mathbf{u_2})]_\gamma$$

¿From the definition of the linear transformation T,

$$T(\mathbf{u_1}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{u_1}, \quad T(\mathbf{u_2}) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{u_2}$$

Hence

$$[T(\mathbf{u_1})]_\gamma = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad [T(\mathbf{e_2})]_\gamma = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

So

$$[T]_\beta^\beta = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

This matrix is 'simpler' in the sense that diagonal matrices enjoy special properties that more general matrices do not.

Much research has been devoted to determining the 'simplest possible form' that can be obtained for the matrix of a linear operator $T : V \to V$, by choosing the basis appropriately. This problem can be attacked by first finding a matrix for T relative to *any* basis, say a standard basis, where applicable, then changing the basis in a manner that simplifies the matrix. Before pursuing this idea further, it is necessary to grasp the theorem below. It gives a useful alternative viewpoint about change of basis matrices; it shows that the transition matrix form a basis β to γ can be regarded as the matrix of transformation of the identity operator.

Theorem 6.11 *If β and γ are bases for a finite-dimensional vector space V, and if $I : V \to V$ is the identity operator, then $[I]_\beta^\gamma$ is the transition matrix form β to γ.*

Proof: Suppose that $\beta = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ and $\gamma = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ are bases for V. Using the fact that $I(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$, it follows that

$$\begin{aligned} [I]_\beta^\gamma &= [[I(\mathbf{u_1})]_\gamma \mid [I(\mathbf{u_2})]_\gamma \mid [I(\mathbf{u_n})]_\gamma] \\ &= [[\mathbf{u_1}]_\gamma \mid [\mathbf{u_2}]_\gamma \mid [\mathbf{u_n}]_\gamma] \end{aligned}$$

which by Theorem 3.8 is the change of basis matrix from β to γ.

The groundwork has been laid to consider the main problem in this section.

Problem: If β and γ are two bases for a finite-dimensional vector space V, and if $T : V \to V$ is a linear operator, what relationship, if any, exists between the matrices $[T]_\beta^\beta$ and $[T]_\gamma^\gamma$?

The answer to this question can be obtained by considering the composition of three linear operators. Consider a vector $\mathbf{v} \in V$. Let the vector $\mathbf{v}$ be mapped into itself by the identity operator, then let $\mathbf{v}$ be mapped into $T(\mathbf{v})$ by T, then let $T(\mathbf{v})$ be mapped into itself by the identity operator. All four vector spaces involved in the composition are the same (namely V); however, the bases for the spaces vary. Since the starting vector is $\mathbf{v}$ and the final vector is $T(\mathbf{v}$, the composition is the same as T; that is,

$$T = I \circ T \circ I$$

If the first and last vector spaces are assigned the basis γ and the middle two spaces are assigned the basis β, then it follows from the previous statement $T = I \circ T \circ I$, that

$$[T]_\gamma^\gamma = [I \circ T \circ I]_\gamma^\gamma = [I]_\beta^\gamma [T]_\beta^\beta [I]_\gamma^\beta$$

But it follows from Theorem 7.1 that $[I]_\gamma^\beta$ is the change of basis matrix from γ to β and consequently $[I]_\beta^\gamma$ is the change of basis matrix from β to γ. Thus, let $P = [I]_\gamma^\beta$, then $P^{-1} = [I]_\beta^\gamma$ and hence it can be written that

$$[T]_\gamma^\gamma = P^{-1}[T]_\beta^\beta P$$

This is all summarised in the following theorem.

Theorem 6.12 *Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let β and γ be bases for V. Then*

$$[T]_\gamma^\gamma = P^{-1}[T]_\beta^\beta P \tag{9}$$

where P is the change of basis matrix from γ to β.

Remark: When applying Theorem 7.3, it is easy to forget whether P is the change of basis matrix from β to γ or the change of basis matrix from γ to β. Just remember that in order for $[T]_\beta^\beta$ to operate successfully on a vector $\mathbf{v}$, $\mathbf{v}$ must be in the basis β. Therefore, due to P's positioning in the formula, it must be the change of basis matrix from γ to β.

Example

1. Let $T : \Re^2 \to \Re^2$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$$

Find the matrix of T with respect to the standard basis $\beta = \{\mathbf{e_1}, \mathbf{e_2}\}$ for $\Re^2$, then use Theorem 7.2 to find the matrix of T with respect to the basis $\gamma = \{\mathbf{u_1}, \mathbf{u_2}\}$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

It was shown earlier that

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

To find $[T]_\gamma^\gamma$ from (9), requires the change of basis matrix P, where

$$P = [I]_\gamma^\beta = [[\mathbf{u_1}]_\beta \mid \mathbf{u_2}]_\gamma]$$

By inspection

$$\begin{aligned} \mathbf{u_1} &= \mathbf{e_1} + \mathbf{e_2} \\ \mathbf{u_2} &= \mathbf{e_1} + 2\mathbf{e_2} \end{aligned}$$

so that

$$[\mathbf{u_1}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad [\mathbf{u_2}]^\beta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Thus the transition matrix from γ to β is

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

It is clear that

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

so that by Theorem 7.2 the matrix of T relative to the basis γ is

$$[T]_\gamma^\gamma = P^{-1}[T]_\beta^\beta P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

which agrees with the previous result.

The relationship in (9) is of such importance that there is some terminology associated with it.

Definitions

- If A and B are square matrices, it is said that B **is similar to** A if there is an invertible matrix P such that $B = P^{-1}AP$.

It follows from the definition that *two matrices representing, the same linear operator $T : V \to V$ with respect to different bases are similar.* The following example demonstrates just this phenomenon. It helps to remember that a change of basis in $\Re^2$ or $\Re^3$ really just amounts to a change in the coordinate system. Note when reading the following example how the choice of the coordinate system can greatly simplify the form of the matrix of the transformation.

Example

1. Let l be the line in the $x - y$ plane that passes through the origin and makes an angle θ with the positive x-axis, where $0 \leq \theta \leq \pi$. Let $T : \Re^2 \to \Re^2$ be the linear operator that maps each vector into its reflection about the line l.

   (a) Find the standard matrix for T.

   (b) Find the reflection of the vector $\mathbf{x} = (1, 2)$ about the line l through the origin that makes an angle of $\theta = \frac{\pi}{6}$ with the positive x-axis.

   Solution:

   (a) This problem could be solved by trying to construct the standard matrix from the formula

   $$[T]_\beta^\beta = [T(\mathbf{e_1}) \mid T(\mathbf{e_2})]$$

   where $\beta = \{\mathbf{e_1}, \mathbf{e_2}\}$ is the standard basis for $\Re^2$. However, it is easier to use a different strategy: Instead of finding $[T]_\beta^\beta$ directly, it is simpler to first find the matrix $[T]_\gamma^\gamma$ where

   $$\gamma = \{\mathbf{u_1}, \mathbf{u_2}\}$$

is the basis of a unit vector $\mathbf{u_1}$ along l and a unit vector $\mathbf{u_2}$ perpendicular to l. Once $[T]_\gamma^\gamma$ has been found, it is simply a matter of using Theorem 7.2 to perform a change of basis to find $[T]_\beta^\beta$. The computations are as follows:

$$T(\mathbf{u_1}) = \mathbf{u_1} \text{ and } T(\mathbf{u_2}) = -\mathbf{u_2}$$

so

$$[T(\mathbf{u_1})]_\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } [T(\mathbf{u_2})]_\gamma = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Thus,

$$[T]_\gamma^\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It is clear from the geometry of the problem that

$$[\mathbf{u_1}]_\beta = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } [\mathbf{u_2}]_\beta = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Therefore the change of basis matrix from β to γ is

$$P = \begin{bmatrix} [\mathbf{u_1}]_\gamma & | & [\mathbf{u_2}]_\gamma \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

It follows from formula (9) that

$$[T]_\beta^\beta = P[T]_\gamma^\gamma P^{-1}$$

Thus, the standard matrix for T is

$$\begin{aligned} [T]_\beta^\beta = P[T]_\gamma^\gamma P^{-1} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \end{aligned}$$

(b) It follows from part (a) that the formula for T in matrix notation is

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Substituting $\theta = \frac{\pi}{6}$ in this formula yields

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \sqrt{3} \\ \frac{\sqrt{3}}{2} - 1 \end{bmatrix}$$

Thus, $T(1,2) = (\frac{1}{2} + \sqrt{3}, \frac{\sqrt{3}}{2} - 1)$.

It has already been stated that the selection of the correct basis can result in a simpler form of the matrix of linear transformation $T : V \to V$. In the previous example, the selection of the basis γ resulted in a diagonal matrix of transformation. Given the computational advantages of diagonal matrices (refer to section 1.6.1), it is important to find ways (if possible) to choose a basis for V such that the transformation matrix is diagonal. This problem can be stated in another way. Given an $n \times n$ matrix of transformation $[T]_\beta^\beta$, what change of basis matrix P is required such that $P^{-1}[T]_\beta^\beta P$ is a diagonal matrix. This very problem is attacked in the next section.

7 EIGENVALUES AND EIGENVECTORS

This section will be concerned with the following two problems.

1. **The Diagonalization Problem (matrix form).** Given an $n \times n$ matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix? Or to state it in another way: If A is an $n \times n$ matrix, does there exist a diagonal matrix D such that A is similar to D?

2. **The Eigenvalue Problem:** If A is an $n \times n$ matrix, are there nonzero vectors $\mathbf{x}$ in $\Re^n$ such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

These two problems are very closely related. In fact, once the eigenvalue problem is solved it leads to the solution of the diagonalization problem. Hence it is first of all necessary to consider the eigenvalue problem.

7.1 The Eigenvalue Problem

Definition

- If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\Re^n$ is called an **eigenvector** of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,
$$A\mathbf{x} = \lambda\mathbf{x}$$
for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\mathbf{x}$ is said to be an eigenvector of A **corresponding** to λ.

Eigenvalues and eigenvectors have many important applications, some of which are given at the end of section 6. For now, however, only a geometric interpretation will be given of the problem in $\Re^2$. If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is an eigenvector of A corresponding to λ, then multiplication of $\mathbf{x}$ by the matrix A produces a vector $\lambda\mathbf{x}$ that is parallel to $\mathbf{x}$.
Examples - *VERIFYING EIGENVALUES AND EIGENVECTORS*

1. For the matrix
$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$
verify that $\mathbf{x_1} = (1,0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$, and $\mathbf{x_2} = (0,1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

   Multiplying A by $\mathbf{x_1}$ produces
$$A\mathbf{x_1} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\mathbf{x_1} = \lambda_1\mathbf{x_1}$$
   Thus $\mathbf{x_1} = (1,0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$.

   Similarly, multiplying A by $\mathbf{x_2}$ produces
$$A\mathbf{x_2} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1\mathbf{x_2} = \lambda_2\mathbf{x_2}$$
   Thus $\mathbf{x_2} = (0,1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

2. For the matrix
$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

verify that $\mathbf{x_1} = (-3,-1,1)$ and $\mathbf{x_2} = (1,0,0)$ are eigenvectors of A and find their corresponding eigenvalues.

Multiplying A by $\mathbf{x_1}$ produces

$$A\mathbf{x_1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = 0\mathbf{x_1} = \lambda_1 \mathbf{x_1}$$

Thus $\mathbf{x_1} = (-3,-1,1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$.

Similarly, multiplying A by $\mathbf{x_2}$ produces

$$A\mathbf{x_1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\mathbf{x_2} = \lambda_2 \mathbf{x_2}$$

Thus $\mathbf{x_2} = (1,0,0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 1$.

7.1.1 Determining the Eigenvalues

It is all very well to verify eigenvalues and eigenvectors, but what if they are not known to verify. The next theorem characterises the eigenvalues of a matrix.

Theorem 7.1 *Let A be $n \times n$, $\lambda \in \Re^n$. Then λ is an eigenvalue of A if and only if it is a solution to the equation*

$$det(\lambda I_n - A)$$

Proof: Assume $A\mathbf{x} = \lambda\mathbf{x},\ \mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \in \Re^n$, then

$$A\mathbf{x} = \lambda\mathbf{x} = I_n\lambda\mathbf{x} = \lambda I_n \mathbf{x}$$

therefore it can be stated that

$$\begin{aligned} \lambda I_n\mathbf{x} - A\mathbf{x} &= \mathbf{0} \\ (\lambda I_n - A)\mathbf{x} &= \mathbf{0} \end{aligned} \tag{10}$$

Due to the fact that $\mathbf{x} \neq \mathbf{0}$, equation (10) results in the condition that the matrix $(\lambda I_n - A)$ cannot have an inverse (otherwise $\mathbf{x} = \mathbf{0}$ would be the one and only solution). Thus the following condition follows:

$$det(\lambda I_n - A) = 0$$

Therefore the values of λ that satisfy this condition are the eigenvalues of the matrix A.

Remark: Note that $\det(A - \lambda I_n)$ also completely characterises the set of eigenvalues.

Definitions

- The equation $\det(\lambda I_n - A) = 0$ is called the **characteristic equation** of A.
- The polynomial $\det(xI_n - A)$ is called the **characteristic polynomial** of A, and is denoted by $ch_A(x)$.
- If λ is an eigenvalue of A and

  $$ch_A(x) = (x-\lambda)^t \cdot g(x),\ g(\lambda) \neq 0$$

  then t is called the **algebraic multiplicity** of λ and is denoted by $a_A(\lambda)$.

Examples

1. Let A be the general $n \times n$ matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

   Then the characteristic polynomial of A is given by

$$ch_A(x) = det(xI_n - A) = \begin{vmatrix} x - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & x - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & x - a_{nn} \end{vmatrix}$$

2. Let A be the 2×2 matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

   then

$$ch_A(x) = det(xI_n - A) = \begin{vmatrix} x - 2 & -1 \\ -1 & x - 2 \end{vmatrix} = (x-2)^2 - 1 = (x-1)(x-3)$$

   So the eigenvalues are $\lambda_1 = 1, \lambda_2 = 3$. Note also that $a_A(1) = 1$ and $a_A(3) = 1$.

3. let A be the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

   then, finding the eigenvalues of A

$$ch_A(x) = det(xI_n - A) = \begin{vmatrix} x-1 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & x-4 \end{vmatrix}$$

$$= (x-1)(x-2)(x-3)(x-4)$$

   So the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 4$. Note also that $a_A(1) = 1$, $a_A(2) = 1$, $a_A(3) = 1$ and $a_A(4) = 1$.

   Remark: The previous problem reveals another property of diagonal matrices. The eigenvalues of a diagonal matrix are always equal to the entries on the diagonal. This is easily proven and is left as an exercise for the reader.

7.1.2 Determining the Eigenspace

The following theorem characterises the eigenvectors of a matrix A corresponding to an eigenvalue λ.

Theorem 7.2 *Let A be an $n \times n$ matrix, then the eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$.*

The proof for this theorem is trivial and is left for the reader as an exercise.

Although only one eigenvector for each eigenvalue was listed in the previous two examples, each of the four eigenvalues in those problems have an infinite number of eigenvectors. For instance, in Example 1 the vectors (2,0) and (-3,0) are eigenvectors of A corresponding to the eigenvalue 2. In fact, if A is an $n \times n$ matrix with an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$, then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of A. This may be seen by letting c be a nonzero scalar, which then produces

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x})$$

It is also true that if $\mathbf{x_1}$ and $\mathbf{x_2}$ are eigenvectors to the *same* eigenvalue λ, then their sum is also an eigenvector corresponding to λ, because

$$A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = \lambda\mathbf{x_1} + \lambda\mathbf{x_2} = \lambda(\mathbf{x_1} + \mathbf{x_2})$$

which shows that the set of all eigenvectors of a given eigenvalue λ, along with the zero vector form a subspace of $\Re^n$. This subspace can be described quite simply and is given a special name. The subspace and associated terms are defined below.

Definitions

- If λ is an eigenvalue of A, the subspace $N(\lambda I_n - A)$ is called the **eigenspace** of A corresponding to λ. It is written $E_A(\lambda) = N(\lambda I_n - A)$.
- The dimension of $E_A(\lambda)$ is called the **geometric multiplicity** of λ and is denoted by $g_A(\lambda) =$ nullity$(\lambda I_n - A)$.

Examples - *FINDING EIGENVALUES AND EIGENVECTORS*

1. Find the eigenvalues and the eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

The characteristic equation of A is given by

$$\begin{aligned} |\lambda I_2 - A| &= \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} \\ &= (\lambda - 2)(\lambda + 5) + 12 \\ &= \lambda^2 + 3\lambda - 10 + 12 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 2)(\lambda + 1) = 0, \end{aligned}$$

which gives $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues of A. Both eigenvalues have an algebraic multiplicity of one. To find the corresponding eigenvectors, Gaussian elimination is used to solve the homogeneous linear system given by $(\lambda I_2 - A)\mathbf{x} = \mathbf{0}$ twice: first for $\lambda = \lambda_1 = -1$, and then for $\lambda = \lambda_2 = -2$. For $\lambda_1 = -1$, the coefficient matrix is

$$(-1)I_2 - A = \begin{bmatrix} -1-2 & 12 \\ -1 & -1+5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

Therefore $x_1 - 4x_2 = 0$. Letting $x_2 = t$, it can be concluded that every eigenvector of λ_1 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ t \neq 0$$

Therefore it can be stated that

$$E_A(-1) = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\} \text{ and } g_A(-1) = 1$$

For $\lambda_2 = -2$, one has

$$(-2)I_2 - A = \begin{bmatrix} -2-2 & 12 \\ -1 & -2+5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

Therefore $x_1 - 3x_2 = 0$. Letting $x_2 = t$, it can be concluded that every eigenvector of λ_2 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ t \neq 0$$

Therefore it can be stated that

$$E_A(-2) = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \text{ and } g_A(-2) = 1$$

It is recommended for the reader to try checking that $A\mathbf{x} = \lambda\mathbf{x}$ for the eigenvalues and eigenvectors found in this example.

2. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Also determine the geometric multiplicity of each eigenvalue.

The characteristic equation of A is

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{vmatrix} = (\lambda-2)^3 = 0.$$

Thus the only eigenvalue is $\lambda = 2$. Note that $a_A(2) = 3$. To find the eigenvectors corresponding to $\lambda = 2$, the homogeneous linear system represented by $(2I_n - A)\mathbf{x} = \mathbf{0}$ must be solved.

$$2I_3 - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$. Therefore, using the parameters $s = x_1$ and $t = x_3$, the eigenvectors of $\lambda = 2$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Where s and t are not both zero. Therefore it can be stated that

$$E_A(2) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \text{ and } g_A(2) = 2$$

Summary - *FINDING EIGENVALUES AND EIGENVECTORS*

Let A be an $n \times n$ matrix.

1. Form the characteristic equation $|\lambda I_n - A| = 0$. It will be a polynomial equation of degree n in the variable λ.
2. Find the roots of the characteristic equation. These are the eigenvalues of A.
3. For each eigenvalue λ_i, find the eigenvectors corresponding to λ_i by solving the homogeneous system $(\lambda_i I_n - A)\mathbf{x} = \mathbf{0}$. This requires row-reducing an $n \times n$ matrix. The resulting reduced row-echelon form must have at least one row of zeros.

7.1.3 Eigenvalues and Eigenvectors of Linear Transformations

This section began with the definition of eigenvalues and eigenvectors in terms of matrices. However, these definitions could have just as easily been given in terms of linear transformations. A number λ is called an **eigenvalue** of a linear transformation $T : V \to V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the **eigenspace** of λ.

Consider the linear transformation $T : \Re^3 \to \Re^3$, whose matrix relative to the standard basis β is

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

It can be shown that the matrix of T relative to the basis $\gamma = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})\}$ where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is the diagonal matrix

$$[T]_\gamma^\gamma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The question has been stated twice before: 'For a given transformation T, how can one know what basis to choose so that the corresponding matrix of transformation is diagonal?' One may be surprised to find that eigenvalues and eigenvectors provide the answers. Consider the matrix of transformation $[T]_\beta^\beta$. Finding the eigenvalues and eigenvectors of

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The eigenvalues can be found by considering the characteristic equation which is given by

$$\begin{aligned} |\lambda I_3 - A| &= \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} \\ &= (\lambda + 2)[(\lambda - 1)^2 - 9] \\ &= (\lambda + 2)(\lambda^2 - 2\lambda - 8) = (\lambda + 2)^2(\lambda - 4) \end{aligned}$$

the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. The eigenspaces for these two eigenvalues are as follows.

$$E_A(4) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$E_A(-2) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Two very interesting points should be observed.

1. Let $T : \Re^3 \to \Re^3$ be the linear transformation whose standard matrix is A, and let γ be the basis of $\Re^3$ made up by the three linearly independent eigenvectors just found of A. Then $[T]_\gamma^\gamma$, as stated earlier is diagonal

$$[T]_\gamma^\gamma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

2. The main diagonal entries of the matrix $[T]_\gamma^\gamma$ are the eigenvalues of A.

The next section formalizes these two results and gives the necessary criterion that ensure that a matrix A is similar to a diagonal matrix.

7.2 Diagonalization

The first objective in this section is to show the equivalence (that was suggested at in the previous section) of the following two problems, which on the surface seem quite different.
The Eigenvector Problem: Given an $n \times n$ matrix A, does there exist a basis for $\Re^n$ consisting of eigenvectors of A?
The Diagonalization problem (Matrix Form): Given an $n \times n$ matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix; the matrix P is said to **diagonalize** A.
The latter problem suggests the following terminology
Definition

- A square matrix A is called **diagonalizable** if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix; the matrix P is said to **diagonalize** A. Or in other words, if a square matrix A is similar to a diagonal matrix D then A is said to be **diagonalizable**.

The following theorem shows that the eigenvector problem and the diagonalization problem are equivalent.

Theorem 7.3 *If A is an $n \times n$ matrix, then the following are equivalent.*

(a) A is diagonalizable.

(b) A has n linearly independent eigenvectors.

Proof:

$(a) \Rightarrow (b)$ Since A is assumed diagonalizable, there is an invertible matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

such that $P^{-1}AP = D$, where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

It follows from the formula $P^{-1}AP = D$ that $AP = PD$; that is,

$$\begin{aligned} AP &= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix} \end{aligned}$$

If we now let $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ denote the column vectors of P, then from above the successive columns of AP are $\lambda_1\mathbf{p_1}, \lambda_2\mathbf{p_2}, \ldots, \lambda_n\mathbf{p_n}$. Thus, we must have

$$A\mathbf{p_1} = \lambda_1\mathbf{p_1},\ A\mathbf{p_2} = \lambda_2\mathbf{p_2},\ \ldots,\ A\mathbf{p_n} = \lambda_n \tag{11}$$

Since P is invertible, its column vectors are all nonzero; thus, it follows from (11) that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A, and $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ are the corresponding eigenvectors. Since P is invertible, it follows from Theorem 5.10 that $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ are linearly independent. Thus, A has n linearly independent eigenvectors which thus form a basis for $\Re^n$.

$(b) \Rightarrow (a)$ Assume that A has n linearly independent eigenvectors, $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

be the matrix whose column vectors are $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$. The column vectors of the product AP are

$$A\mathbf{p_1},\ A\mathbf{p_2},\ \ldots, A\mathbf{p_n}$$

But

$$A\mathbf{p_1} = \lambda_1\mathbf{p_1},\ A\mathbf{p_2} = \lambda_2\mathbf{p_2},\ \ldots,\ A\mathbf{p_n} = \lambda_n\mathbf{p_n}$$

so that

$$\begin{aligned} AP &= \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix} \\ &= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD \end{aligned}$$

where D is the diagonal matrix having the eigenvalues of $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the main diagonal. Since the column vectors of P are linearly independent, P is invertible; thus it can be written that $P^{-1}AP = D$; that is, A is diagonalizable.

7.2.1 Procedure for Diagonalizing a Matrix

Theorem 7.3 guarantees than an $n \times n$ matrix with n linearly independent eigenvectors is diagonalizable, and the proof provides the following method for diagonalizing A.

Step 1. Find n linearly independent eigenvectors of A, say, $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$.

Step 2. Form the matrix P having $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ as its column vectors.

Step 3. The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to $\mathbf{p_i}$, for $i = 1, 2, \ldots, n$.

In order to carry out Step 1 of this procedure, one first needs a way of determining whether a given $n \times n$ matrix A has n linearly independent eigenvectors, and then one needs a method for finding them. One can address both problems at once by finding bases for the eigenspaces of A. Later in this section it shall be shown that those basis vectors, as a combined set, are linearly independent, so that if there is a total of n such vectors, then A is diagonalizable, and the n basis vectors can be used as the column vectors of the diagonalizing matrix P. If there are fewer than n basis vectors, the A is not diagonalizable.
Examples: *DIAGONALIZING MATRICES*

1. Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

It is left for the reader to verify that the characteristic equation of A is given by

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and the corresponding eigenspaces have the bases shown below

$$\lambda = 2: \quad \mathbf{p_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\lambda = 1: \quad \mathbf{p_3} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix A is diagonalizable and

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes A. As a check, it can be observed that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There is no preferred order for the columns of P. Since the i^{th} diagonal entry of $P^{-1}AP$ is an eigenvalue ofr the i^{th} column vector of P, changing the order of the columns of P just changes the order of the eigenvalues on the diagonal of $P^{-1}AP$. Thus, had it been written

$$P = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then one would have obtained

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

The characteristic equation of A is given by

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2 = 0$$

Thus the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$. It is left for the student to verify that the bases for the eigenspaces of λ_1 and λ_2 are:

$$\begin{aligned} E_A(1) &= \mathbf{p_1} = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix} \\ E_A(2) &= \mathbf{p_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Since A is a 3×3 matrix and there are only two basis vectors in total, A is not diagonalizable.

There was an assumption made in the previous examples, that the column vectors of P, which are made up of basis vectors from the various eigenspaces of A, are linearly independent. The following theorem addresses this issue.

Theorem 7.4 *If* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ *are eigenvectors of* A *corresponding to distinct eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *then* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ *is a linearly independent set.*

Proof: Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. If one assumes that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are linearly dependent, then a contradiction can be obtained. It can then be concluded that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are linearly independent.

Since an eigenvector is nonzero by definition, $\{\mathbf{v_1}\}$ is linearly independent. Let r be the largest integer such that $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is linearly independent. Assuming that $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is linearly dependent, r satisfies $1 \leq r < k$. Moreover, by definition of r, $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{r+1}}\}$ is linearly dependent. Thus, these are scalars $c_1, c_2, \ldots, c_{r+1}$, not all zero, such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_{r+1}\mathbf{v_{r+1}} = \mathbf{0} \tag{12}$$

Multiplying both sides of (12) by A and using

$$A\mathbf{v_1} = \lambda_1\mathbf{v_1},\ A\mathbf{v_2} = \lambda_2\mathbf{v_2},\ \ldots,\ A\mathbf{v_{r+1}} = \lambda_{r+1}\mathbf{v_{r+1}}$$

it is obtained that

$$c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} + \cdots + c_{r+1}\lambda_{r+1}\mathbf{v_{r+1}} = \mathbf{0} \tag{13}$$

Multiplying both sides of (12) by λ_{r+1} and subtracting the resulting equation from (13) yields

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{v_1} + c_2(\lambda_2 - \lambda_{r+1})\mathbf{v_2} + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{v_r} = \mathbf{0}$$

Since $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a linearly independent set, this equation implies that

$$c_1(\lambda_1 - \lambda_{r+1}) = c_2(\lambda_2 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

and since $\lambda_1, \lambda_2, \ldots, \lambda_{r+1}$ are distinct, it follows that

$$c_1 = c_2 = \cdots = c_r = 0 \tag{14}$$

Substituting these values into (12) yields

$$c_{r+1}\mathbf{v_{r+1}} = \mathbf{0}$$

Since the eigenvector $\mathbf{v_{r+1}}$ is nonzero, it follows that

$$c_{r+1} = 0 \tag{15}$$

Equations (14) and (15) contradict the fact that $c_1, c_2, \ldots, c_{r+1}$ are not all zero; this completes the proof.

7.2.2 Conditions on Diagonalability

The previous theorem leads into some results regarding the diagonalability of matrices. The next theorem is a direct result of Theorem 7.4.

Theorem 7.5 *If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.*

Proof: If $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ are eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then by Theorem 7.4 $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ are linearly independent. Thus, A is diagonalizable by Theorem 7.3.

Examples

1. It is known that the matrix

   $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

   has three distinct eigenvalues, $\lambda_1 = 4, \lambda_2 = 2+\sqrt{3}, \lambda_3 = 2-\sqrt{3}$. Therefore, A is diagonalizable. Further,

   $$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$$

   for some invertible matrix P. If desired, the matrix P can be found using the method outlined previously.

2. The eigenvalues of a triangular matrix are the entries on its main diagonal. Thus, a triangular matrix with distince entries on the main diagonal is diagonalizable. For example,

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

is a diagonalizable matrix

There is one final result that is well worth mentioning. Although the proof of this theorem is well beyond the scope of this course even an elementary understanding of this theorem will lead to a fuller understanding of diagonalability.

Theorem 7.6 *If A is a square matrix, then:*

(a) For every eigenvalue of A the geometric multiplicity is less than or equal to the algebraic multiplicity.

(b) A is diagonalizable if and only if the geometric multiplicity is equal to the algebraic multiplicity for every eigenvalue.

Once a sound understanding of all the previous theorems has been maintained it is easy to see how the second part of Theorem 7.6 follows directly from part (a).
Example

1. Consider the 3×3 matrix

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

A is a lower triangular matrix, hence the eigenvalues are the entries on the main diagonal. Therefore $\lambda_1 = 5$ and $\lambda_2 = 4$. Note also that $a_A(5) = 2$ (it occurs on the main diagonal twice), and $a_A(4) = 1$. Consider the eigenspace corresponding to the eigenvalue $\lambda_1 = 5$. It can be shown by the usual method that

$$E_A(5) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Thus $g_A(5) = 1$. Note that $g_A(5) = 1 \leq 2 = a_A(5)$ and hence by Theorem 7.6(b) the matrix A is not diagonalizable. One does not even have to check for the eigenspace of $\lambda_2 = 4$. This theorem becomes extremely handy in matrices larger than 3×3.

7.3 Orthogonal Diagonalization

For most matrices, much of the diagonalization process must be completed before it can be determined whether or not the matrix is diagonalizable. One exception is the triangular matrix with distinct entries on the main diagonal. This section will study another type of matrix that is guaranteed to be diagonalizable: a **symmetric** matrix. It well be shown that these **symmetric** matrices are diagonalized by matrix consisting of an orthogonal set of eigenvectors. Before establishing this, however, it is important to grasp some theory regarding **symmetric** and **orthogonal** matrices.

7.3.1 Symmetric Matrices

Definitions

- A square matrix A is **symmetric** if
$$A = A^T$$

Examples: *SYMMETRIC MATRICES AND NON-SYMMETRIC MATRICES*

1. The below matrices are symmetric
$$\begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & 2 \\ 3 & 7 & 6 \\ 2 & 6 & 1 \end{bmatrix}$$

2. The below matrices are almost symmetric, but almost is not good enough.
$$\begin{bmatrix} 3 & 4 & 0 \\ 4 & 2 & -1 \\ 0 & 1 & 7 \end{bmatrix} \quad \begin{bmatrix} 2 & 5 & 5 \\ 5 & 2 & 8 \\ 6 & 8 & 2 \end{bmatrix}$$

Unlike non-symmetric matrices, symmetric matrices have a series of very useful properties regarding their eigenvalues which guarantee their diagonalability. These properties are summarised in the following theorem.

Theorem 7.7 *If A is an $n \times n$ symmetric matrix, then the following properties are true.*

(a) All eigenvalues of A are real.

(b) If λ is an eigenvalue of A, then $a_A(\lambda) = g_A(\lambda)$ for all λ.

Hence theorem 7.6 guarantees the diagonalability of symmetric matrices.

The proof of this theorem requires a knowledge of complex vector spaces and is therefore omitted. However, the following two examples demonstrate parts (a) and (b) of the theorem respectively.

Examples

1. It can be proved that a 2×2 symmetric matrix always has real eigenvalues and is always diagonalizable. Consider the general 2×2 symmetric matrix A where
$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$
The characteristic polynomial of A is given by
$$|\lambda I_2 - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2$$
As a quadratic in λ, this polynomial has a discriminant of
$$\begin{aligned} (a+b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab - 4c^2 \\ &= a^2 - 2ab + b^2 - 4c^2 \\ &= (a-b)^2 + 4c^2 \end{aligned}$$

Since this discriminant is the sum of two squares, it must be either zero or positive. Therefore the eigenvalues are always real. Also A must always be diagonalizable. Consider the two cases. If $(a-b)^2 - 4c^2 = 0$, then $a = b$ and $c = 0$, which implies that A is already diagonal. That is,

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

On the other hand, if $(a-b)^2 + 4c^2 > 0$, then by the Quadratic Formula the characteristic polynomial of A has two distinct real roots, which implies that A has two distinct eigenvalues. Thus A is diagonalizable in this case also.

2. Find the eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

and determine the dimensions of the corresponding eigenspaces.

The characteristic polynomial for A is given by

$$|\lambda I_4 - A| = \left| \begin{array}{cccc} \lambda - 1 & 2 & 0 & 0 \\ 2 & \lambda - 1 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 2 \\ 0 & 0 & 2 & \lambda - 1 \end{array} \right] = (\lambda + 1)^2(\lambda - 3)^2$$

Thus the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$. Since each of these eigenvalues has a algebraic multiplicity of 2, it is known form Theorem 7.7(b) that the corresponding eigenspaces also have a dimension of 2. Specifically, the eigenspace of $\lambda_1 = -1$ has a basis of

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and the eigenspace of $\lambda_2 = 3$ has a basis of

$$\gamma = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and hence the matrix is diagonalizable.

7.3.2 Orthogonal Matrices

Recall that to diagonalize a square matrix A, an invertible matrix P is needed such that $P^{-1}AP$ is diagonal. For symmetric matrices, it shall be shown that the matrix P can be chosen to have the special property that $P^{-1} = P^T$. This unusual matrix property is defined as follows
Definitions

- A square matrix P is called **orthogonal** if it is invertible and

$$P^{-1} = P^T$$

Example: *AN ORTHOGONAL MATRIX*

1. The matrix
$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
is orthogonal since
$$P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Recall also that two vectors $\mathbf{p_1}$ and $\mathbf{p_2}$, are orthogonal in $\Re^n$ if and only if $\mathbf{p_1} \cdot \mathbf{p_2} = 0$, and they are orthonormal if in addition, $||\mathbf{p_i}|| = 1$ for $i = 1, 2$. In the previous example, it can be observed that the columns of P form an orthonormal set of vectors. This result suggests the following theorem.

Theorem 7.8 *An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthonormal set.*

Proof: Suppose that the column vectors of P form an orthonormal set:
$$\begin{aligned} P &= [\mathbf{p_1} \,|\, \mathbf{p_2} \,|\, \cdots \,|\, \mathbf{p_n}] \\ &= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \end{aligned}$$

Then the product P^TP has the form
$$\begin{aligned} P^TP &= \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p_1} \cdot \mathbf{p_1} & \mathbf{p_1} \cdot \mathbf{p_2} & \cdots & \mathbf{p_1} \cdot \mathbf{p_n} \\ \mathbf{p_2} \cdot \mathbf{p_1} & \mathbf{p_2} \cdot \mathbf{p_2} & \cdots & \mathbf{p_2} \cdot \mathbf{p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p_n} \cdot \mathbf{p_1} & \mathbf{p_n} \cdot \mathbf{p_2} & \cdots & \mathbf{p_n} \cdot \mathbf{p_n} \end{bmatrix} \end{aligned}$$

Since the set $\{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\}$ is orthonormal, one has
$$\mathbf{p_i} \cdot \mathbf{p_j}, \; i \neq j \text{ and } \mathbf{p_i} \cdot \mathbf{p_i} = ||\mathbf{p_i}||^2 = 1$$

Thus the matrix composed of dot products has the form
$$P^TP = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

This implies that $P^T = P^{-1}$, and one concludes that P is orthogonal.

Assume now that P is an $n \times n$ orthogonal matrix. Let P be the matrix
$$P = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix}$$

As P is orthogonal, $P^T = P^{-1}$ and therefore $P^T P = I_n$. It is known that

$$\begin{aligned}
P^T P &= \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{p_1} \cdot \mathbf{p_1} & \mathbf{p_1} \cdot \mathbf{p_2} & \cdots & \mathbf{p_1} \cdot \mathbf{p_n} \\ \mathbf{p_2} \cdot \mathbf{p_1} & \mathbf{p_2} \cdot \mathbf{p_2} & \cdots & \mathbf{p_2} \cdot \mathbf{p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{p_n} \cdot \mathbf{p_1} & \mathbf{p_n} \cdot \mathbf{p_2} & \cdots & \mathbf{p_n} \cdot \mathbf{p_n} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n
\end{aligned}$$

where

$$P = [\mathbf{p_1} \mid \mathbf{p_2} \mid \cdots \mid \mathbf{p_n}]$$

therefore one has that

$$\mathbf{p_i} \cdot \mathbf{p_j},\ i \neq j \text{ and } \mathbf{p_i} \cdot \mathbf{p_i} = \|\mathbf{p_i}\|^2 = 1$$

and hence $\{\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}\}$ form an orthonormal set.

Example: *AN ORTHOGONAL MATRIX*

1. Show that

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

is orthogonal by showing that $P^T P = I_3$. Then show that the column vectors of P form an orthonormal set.

Since

$$P^T P = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

It follows that $P^T = P^{-1}$, and it is concluded that P is orthogonal. Moreover it is clear that the columns of P form an orthonormal set, as letting

$$\mathbf{p_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix},\ \mathbf{p_2} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix},\ \mathbf{p_3} = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix}$$

produces

$$\mathbf{p_1} \cdot \mathbf{p_2} = \mathbf{p_1} \cdot \mathbf{p_3} = \mathbf{p_2} \cdot \mathbf{p_3} = 0$$

and

$$\|\mathbf{p_1}\| = \|\mathbf{p_2}\| = \|\mathbf{p_3}\| = 1$$

Therefore $\{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}\}$ form an orthonormal set. Note that all those calculations take place in the operation $P^T P$.

Before presenting the main result of this section, the following preliminary theorem is given, which states that for symmetric matrices eigenvectors corresponding to distinct eigenvalues are orthogonal.

Theorem 7.9 *Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors $\mathbf{x_1}$ and $\mathbf{x_2}$ are orthogonal.*

Proof: Let λ_1 and λ_2 be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{x_1}$ and $\mathbf{x_2}$. Thus $A\mathbf{x_1} = \lambda_1\mathbf{x_1}$ and $A\mathbf{x_2} = \lambda_2\mathbf{x_2}$. To prove the theorem, it is useful to start with the following matrix form of the dot product.

$$\mathbf{x_1} \cdot \mathbf{x_2} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix} = \mathbf{x_1}^T\mathbf{x_2}$$

Now it can be written

$$\begin{aligned} \lambda_1(\mathbf{x_1} \cdot \mathbf{x_2}) &= (\lambda_1\mathbf{x_1}) \cdot \mathbf{x_2} = (A\mathbf{x_1}) \cdot \mathbf{x_2} = (A\mathbf{x_1})^T\mathbf{x_2} \\ &= (\mathbf{x_1}^T A^T)\mathbf{x_2} = (\mathbf{x_1}^T A)\mathbf{x_2} = \mathbf{x_1}^T(A\mathbf{x_2}) \\ &= \mathbf{x_1}^T(\lambda_2\mathbf{x_2}) = \mathbf{x_1} \cdot (\lambda_2\mathbf{x_2}) = \lambda_2(\mathbf{x_1} \cdot \mathbf{x_2}). \end{aligned}$$

This implies that $(\lambda_1 - \lambda_2)(\mathbf{x_1} \cdot \mathbf{x_2}) = 0$, and because $\lambda_1 \neq \lambda_2$ it follows that $\mathbf{x_1} \cdot \mathbf{x_2} = 0$. Therefore $\mathbf{x_1}$ and $\mathbf{x_2}$ are orthogonal.

7.3.3 Orthogonal Diagonalization

Definition

- A matrix A is said to be **orthogonally diagonalizable** if there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal.

The following important theorem states that the set of orthogonally diagonalizable matrices is precisely the set of symmetric matrices.

Theorem 7.10 *Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable if and only if A is symmetric.*

Proof: The proof of the theorem is straight forward in one direction. That is, if it is assumed that A is orthogonally diagonalizable, then there exists an orthogonal matrix P such that $D = P^{-1}AP$ is diagonal. Moreover, since $P^{-1} = P^T$

$$A = PDP^{-1} = PDP^T,$$

which implies that

$$A^T = (PDP^T)^T = (P^T)^T(PD)^T = PD^TP^T = PDP^T = A$$

therefore A is symmetric.

The proof of the theorem in the other direction is more involved, but it is important because it is constructive. Assume that A is symmetric. If A has an eigenvalue λ of algebraic multiplicity k, then by Theorem 7.7(b), λ has k linearly independent eigenvectors. Through the Gram-Schmidt orthonormalization process, this set of k vectors can be used to form an orthonormal basis of eigenvectors for the eigenspace corresponding to λ. This procedure is repeated for each eigenvalue of A. The collection of all resulting eigenvectors is orthogonal by Theorem 7.9, and it is known from the normalization process that this new collection is also orthonormal. Let P be the matrix whose columns consist of these n orthonormal eigenvectors. By Theorem 7.8, P is an orthogonal matrix. Finally by Theorem 7.3, it can be concluded that $P^{-1}AP$ is diagonal. Hence A is orthogonally diagonalizable.

Example: *WHICH MATRIX IS ORTHOGONALLY DIAGONALIZABLE*

1.

$$A_1 = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 6 & 5 \\ 4 & 5 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 7 \\ -1 & 7 & 6 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

It is known by Theorem 7.10, that the only diagonalizable matrices are the symmetric ones, which in this case is only A_1.

It was mentioned above that the second section of the proof of Theorem 7.10 is *constructive*. That is, it gives steps to follow to obtain an orthogonal matrix P that diagonalizes a symmetric matrix. These steps are summarised as follows.

7.3.4 Procedure for Orthogonal Diagonalization

Let A be an $n \times n$ symmetric matrix.

Step 1. Find all eigenvalues of A and determine the algebraic multiplicity of each.

Step 2. For each eigenvalue of algebraic multiplicity 1, choose a unit eigenvector.

Step 3. For each eigenvalue of algebraic multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. (This is always possible as Theorem 7.7 shows.) If this set is not orthonormal, apply the Gramm-Schmidt orthonormalization process.

Step 4. The composite of steps 2 and 3 produces an orthonormal set of n eigenvectors. Use these eigenvectors to form the colnmns of P. The matrix $P^{-1}AP = P^TAP = D$ will be diagonal. (The main diagonal entries of D are the eigenvalues of A.)

Examples: *ORTHOGONAL DIAGONALIZATION*

1. Find an orthogonal matrix P that orthogonally diagonalizes

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Step 1. The characteristic polynomial of A is

$$\begin{aligned} |\lambda I_2 - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} &= (\lambda+3)(\lambda-3) - 1^2 \\ &= \lambda^2 - 6\lambda + 9 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$

Step 2. Now for each eigenvalue of algebraic multiplicity 1, an eigenvector is found by converting the matrix $\lambda I_2 - A$ to reduced row-echelon form. Note that in this case, both λ_1 and λ_2 have an algebraic multiplicity of 1.

For $\lambda_1 = 4$

$$B = \lambda_1 I_2 - A = 4I_2 - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \text{rref}B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Therefore the eigenspace corresponding to $\lambda_1 = 4$ is spanned by the basis

$$E_A(4) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

For $\lambda_2 = 2$

$$B = \lambda_2 I_2 - A = 2I_2 - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \Rightarrow \text{rref}B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore the eigenspace corresponding to $\lambda_2 = 2$ is spanned by the basis

$$E_A(2) = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

The eigenvectors (1,1) and (1,-1) form an *orthogonal* basis for $\Re^2$. Each of these is normalized to produce an *orthonormal* basis. The new bases for the eigenspaces are given below.

$$E_A(4) = \{\mathbf{p_1}\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \qquad E_A(2) = \{\mathbf{p_2}\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

Step 3. Because each eigenvalue has an algebraic multiplicity of 1, go directly to step 4.

Step 4. Using $\mathbf{p_1}$ and $\mathbf{p_2}$ as column vectors, the matrix P is constructed.

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

P is verified by computing $P^{-1}AP = P^TAP = D$.

$$P^TAP = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

2. Find an orthogonal matrix P that orthogonally diagonalizes

$$A = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$$

Step 1. The characteristic polynomial of A is

$$\begin{aligned} |\lambda I_4 - A| &= \begin{vmatrix} \lambda+7 & -24 & 0 & 0 \\ -24 & \lambda-7 & 0 & 0 \\ 0 & 0 & \lambda+7 & -24 \\ 0 & 0 & -24 & \lambda-7 \end{vmatrix} \\ &= (\lambda+7)(\lambda-7)[(\lambda+7)(\lambda-7)-24^2] \\ &\quad -24^2[(\lambda+7)(\lambda-7)-24^2] \\ &= [(\lambda+7)(\lambda-7)-24^2]^2 \\ &= (\lambda^2-49-576)^2 \\ &= (\lambda^2-625)^2 \\ &= [(\lambda+25)(\lambda-25)]^2 \\ &= (\lambda+25)^2(\lambda-25)^2 \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = 25$ and $\lambda_2 = -25$. Note also that $a_A(25) = a_A(-25) = 2$.

Step 2. Neither of the eigenvalues have an algebraic multiplicity of one. Hence go to step 3.

Step 3. The eigenspaces of each eigenvalue are found by solving the homogeneous system $[\lambda I_4 - A]\mathbf{x} = \mathbf{0}$. This can be done by converting the matrix $\lambda I_4 - A$ to reduced row-echelon form.

For $\lambda_1 = 25$

$$B = \lambda_1 I_4 - A = 25I_4 - A = \begin{bmatrix} 32 & -24 & 0 & 0 \\ -24 & 18 & 0 & 0 \\ 0 & 0 & 32 & -24 \\ 0 & 0 & -24 & 18 \end{bmatrix}$$

$$\Longrightarrow \text{rref}B = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the eigenspace corresponding to $\lambda_1 = 25$ is spanned by the basis

$$E_A(25) = \left\{ \begin{bmatrix} \frac{3}{4} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{3}{4} \\ 1 \end{bmatrix} \right\}$$

For $\lambda_2 = -25$

$$B = \lambda_2 I_4 - A = -25I_4 - A = \begin{bmatrix} -18 & -24 & 0 & 0 \\ -24 & -32 & 0 & 0 \\ 0 & 0 & -18 & -24 \\ 0 & 0 & -24 & -32 \end{bmatrix}$$

$$\Longrightarrow \text{rref}B = \begin{bmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the eigenspace corresponding to $\lambda_2 = -25$ is spanned by the basis

$$E_A(-25) = \left\{ \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ 1 \end{bmatrix} \right\}$$

These two bases are *orthogonal* but not *orthonormal.* Hence an orthonormal bases for each eigenspace is calculated using the Gramm-Schmidt process. The process yields the following two bases

$$E_A(25) = \{\mathbf{p_1}, \mathbf{p_2}\} = \left\{ \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \right\}$$

$$E_A(-25) = \{\mathbf{p_3}, \mathbf{p_4}\} = \left\{ \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \right\}$$

Step 4. Using $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ and $\mathbf{p_4}$ as column vectors, the matrix P is constructed

$$P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \\ 0 & \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

P is verified by computing $P^{-1}AP = P^T AP = D$. It is easily shown that

$$P^{-1}AP = P^T AP = \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & -25 \end{bmatrix}$$

7.4 Application of Eigenvalues and Eigenvectors

The applications of Eigenvalues and Eigenvectors are far two numerous to mention them all. In this brief course on linear algebra there is only sufficient time to mention and summarise three of their most simple applications.

7.4.1 Calculation of Matrix Powers

Let A be an $n \times n$ matrix. To calculate A^m for large m is laborious and time consuming. However if A is diagonalizable this process can be greatly simplified. Consider the following construction.

Let A be an $n \times n$ diagonalable matrix, and let P be an invertible matrix whose column vectors are the eigenvectors of A, then

$$P^{-1}AP = D$$

where D is a diagonal matrix. If $D = P^{-1}AP$ then it follows that

$$A = PDP^{-1}$$

Therefore A^m where $m \geq 1$ is equivalent to

$$(PDP^{-1})^m$$

Consider the special cases for $m = 2$ and $m = 3$

$$\begin{aligned} m = 2 \quad &\rightarrow \quad (PDP^{-1})^2 = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1} \\ m = 3 \quad &\rightarrow \quad (PDP^{-1})^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PDIDIDP^{-1} = PD^3P^{-1} \end{aligned}$$

Therefore in general it can be stated that

$$A^m = (PDP^{-1})^m = PD^mP^{-1}$$

Recall from Section 1.6.1 that if D is the general diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

then

$$D^m = \begin{bmatrix} d_1{}^m & 0 & \dots & 0 \\ 0 & d_2{}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n{}^m \end{bmatrix}$$

Thus calculating the m^{th} power of a matrix can be done by constructing a matrix P such that $P^{-1}AP = D$ where D is diagonal. Then the calculation A^m, which involves m matrix multiplications, reduces to PD^mP^{-1} which, due to the simplicity of the form of D^m, involves only three matrix multiplications.

Remarks: Note that it would be foolish to use this method for values of $m \leq 5$ as there is not a significant reduction in work. Also note that this method is only possible for matrices which are diagonalizable.

Examples: *MATRIX POWERS USING EIGENVALUES AND EIGENVECTORS*

1. Let A be the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

It is left for the reader to show that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. The corresponding eigenspaces are spanned by the bases

$$E_A(1) = \{\mathbf{p_1}\} = \left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \qquad E_A(3) = \{\mathbf{p_2}\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

Using $\mathbf{p_1}$ and $\mathbf{p_2}$ as column vectors, the matrix P is constructed.

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Then $P^{-1}AP = D$ where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Thus it can be stated that

$$\begin{aligned} A^m &= PD^mP^{-1} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 3^m + 1 & 3^m - 1 \\ 3^m - 1 & 3^m + 1 \end{bmatrix} \end{aligned}$$

7.4.2 Systems of Linear Differential Equations

A **system of first-order linear differential equations** has the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

where each y_i is a function of t and $y_i' = dy_i/dt$. Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

then the system can be written in matrix form as

$$\mathbf{y}' = A\mathbf{y}$$

where A is the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Example: *A SIMPLE SYSTEM*

1. Solve the following system of linear differential equations.
   $y_1' = 4y_1$
   $y_2' = -y_2$
   $y_3' = 2y_3$
   ¿From calculus it is known that the solution of the differential equation
   $y' = ky$
   is
   $y = Ce^{kt}$
   Therefore the solution of the given system is
   $y_1 = C_1e^{4t}$
   $y_2 = C_2e^{-t}$
   $y_3 = C_3e^{2t}$

The matrix form of the system of linear differential equations in the previous example is $\mathbf{y}' = A\mathbf{y}$, or

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Thus the coefficients of t in the solutions $y_i = C_ie^{\lambda_i t}$ are given by the eigenvalues of the matrix A.

If A is a diagonal matrix, then the solution of $\mathbf{y}' = A\mathbf{y}$ can be obtained immediately, as in the previous example. If A is not diagonal, then the solution requires a little more work. First it must be attempted to find a matrix P that diagonalizes A. Then the change of variables given by $\mathbf{y} = P\mathbf{w}$ and $\mathbf{y}' = P\mathbf{w}'$ produces

$$P\mathbf{w}' = AP\mathbf{w} \quad \Rightarrow \quad \mathbf{w}' = P^{-1}AP\mathbf{w}$$

where $P^{-1}AP$ is a diagonal matrix. This procedure is demonstrated in next example.
Example: *SOLVING BY DIAGONALIZATION*

1. Solve the following system of linear differential equations.

$$\begin{aligned} y_1' &= 3y_1 + 2y_2 \\ y_2' &= 6y_1 - y_2 \end{aligned}$$

First it is necessary to find a matrix P which diagonalizes the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 5$, with corresponding eigenvectors $\mathbf{v_1} = (1, -3)$ and $\mathbf{v_2} = (1, 1)$. Since the eigenvalues are distinct, it is known that there exists a non-singular matrix P that diagonalizes A. The columns of P are the eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$. That is,

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

The system represented by $\mathbf{w}' = P^{-1}AP\mathbf{w}$ has the following form.

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcr} w_1' & = & -3w_1 \\ w_2' & = & 5w_2 \end{array}$$

The solution to this system of equations is

$w_1 = C_1 e^{-3t}$
$w_2 = C_2 e^{5t}$

To return to the original variables y_1 and y_2, use the substitution $\mathbf{y} = P\mathbf{w}$. Then,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

which implies that the solution is

$$\begin{array}{rcrcr} y_1 & = & w_1 + w_2 & = & C_1 e^{-3t} + C_2 e^{5t} \\ y_2 & = & -3w_1 + w_2 & = & -3C_1 e^{-3t} + C_2 e^{5t} \end{array}$$

7.4.3 Classification of Conics

This application of eigenvectors and eigenvalues is only mentioned briefly. Eigenvectors are found to point in the direction of the major and minor axes of conics. Establishing what the eigenvectors are and then performing a rotation of the standard basis to the eigenvector basis results in a simplified form of equation. Once the equation is expressed in terms of eigenvector coordinates, its classification and sketching are a trivial matter. For a full discussion, see ELEMENTARY LINEAR ALGEBRA by Larson and Edwards.