Tutorial Sheet One

1. Which of the following are linear equations in x_1, x_2 and x_3?

   (a) $x_1 + 2x_2 + 7x_3 = 6$ (b) $x_1x_3 + x_2 = 3$

   (c) $x_1 + 3x_3 = -2x_2 + \frac{1}{5}$ (d) $x_1 = 3\sqrt{x_3} + {x_2}^2$

   (e) $x_1 = x_2$ (f) ${x_1}^2 + {x_2}^2 + 2{x_3}^2 = 3^2$

2. Classify the following systems of equations as non-linear or linear. Further classify the linear systems as non-homogenous or homogenous and form the augmented matrix of the system of linear equations.

   (a)
$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & 0 \\ 3x_1 & + & 4x_2 & = & -1 \\ 2x_1 & - & x_2 & = & 3 \end{array}$$

   (b)
$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 0 \\ 5x_1 & - & 2x_2 & - & 3x_3 & = & 0 \\ -7x_1 & + & x_2 & + & 2x_3 & = & 0 \end{array}$$

   (c)
$$\begin{array}{rcrcrcr} 2x_1 & - & 3{x_2}^2 & + & x_3 & = & 0 \\ -5x_1 & - & 5{x_2}^2 & - & x_3 & = & 0 \\ 3x_1 & + & {x_2}^2 & + & x_3 & = & 0 \end{array}$$

   (d)
$$\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 1 \\ -x_1 & + & 2x_2 & - & x_3 & = & 3 \end{array}$$

3. Using elementary row operations convert the following matrices to reduced row-echelon form

$$(a) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (d) \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ -4 & 0 & 0 \end{bmatrix}$$

4. Solve the following homogenous system by finding the reduced row-echelon form of the coefficient matrix:

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0$$

5. Solve the following homogenous system by finding the reduced row-echelon form

of the coefficient matrix:

$$\begin{aligned} -3x_1 &+ x_2 &+ x_3 &+ x_4 &= 0 \\ x_1 &- 3x_2 &+ x_3 &+ x_4 &= 0 \\ x_1 &+ x_2 &- 3x_3 &+ x_4 &= 0 \\ x_1 &+ x_2 &+ x_3 &- 3x_4 &= 0 \end{aligned}$$

6. Consider the system of equations

$$\begin{aligned} x + y + 2z &= a \\ x \qquad + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

Show that in order for this system to be consistent, a, b and c must satisfy $c = a + b$.

7. For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} x + 2y + 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

8. Let A be $n \times n$, then

   (a) If $A^2 = 0$ prove that A is singular

   (b) If $A^2 = A$ and $A \neq I_n$ prove that A is singular.

9. Calculate A^9, A^T and A^{-1} when

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Tutorial Sheet Two

1. Let V be a real vector space, $\mathbf{p}$ a fixed vector in V. Let us define a new addition and scalar multiplication on V by the formulae

$$\begin{aligned} \mathbf{u} \oplus \mathbf{v} &= \mathbf{u} + \mathbf{v} + \mathbf{p} \\ t \otimes \mathbf{v} &= t\mathbf{v} + (t-1)\mathbf{p} \end{aligned}$$

for all $\mathbf{u}, \mathbf{v} \in V$, $t \in \Re$. Show that V is also a vector space under the new operations, with $-\mathbf{p}$ as the zero vector and $-(\mathbf{v} + \mathbf{2p})$ as the additive inverse of $\mathbf{v}$.

2. Define a new addition on $\Re^2$ by

$$(x_1, y_1) \oplus (x_2, y_2) = ([{x_1}^3 + {x_2}^3]^{\frac{1}{3}}, [{y_1}^3 + {y_2}^3]^{\frac{1}{3}})$$

Show that with this new addition and the usual scalar multiplication, $(0, 0)$ is still the additive identity and $(-x, -y)$ the additive inverse of (x, y). Also verify that all the vector space axioms hold, apart from the axiom

$$(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$$

3. If U_1 and U_2 are subspaces of a vector space V and $U_1 \cup U_2$ is also a subspace of V, prove that $U_1 \subset U_2$ or $U_2 \subset U_1$. (Hint: Prove that $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$ implies $U_1 \cup U_2$ is not closed under addition.)

4. If U and V are subspaces of a vector space W, prove that the subset of W defined by $U + V = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a supspace of W.

5. Let $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}$ and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$ belong to V. Let

$$U_1 = \text{span}(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m})$$

$$U_2 = \text{span}(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$$

prove that

$$U_1 + U_2 = \text{span}(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_m}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$$

6. If $\mathbf{u}$,$\mathbf{v}$,$\mathbf{w}$ belong to the real vector space V, prove that

$$\text{span}(\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}) = \text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$$

7. U and V are subspaces of $\Re^3$ defined by

$$U = \{(x, y, z) \mid x + y + z = 0\} \quad \text{and} \quad V = \{(x, y, z) \mid x - y - z = 0\}$$

Find spanning families for U and V and prove that $U + V = \Re^3$.

8. Which of the following subsets of $\Re^2$ are subspaces of $\Re^2$?

(a) $\{(x, y) \mid x = 3y\}$

(b) $\{(x, y) \mid x^2 = y^2\}$

(c) $\{(x, y) \mid x + y = 1\}$

(d) $\{(x, y) \mid x \geq 0 \text{ and } y \geq 0\}$

9. Let $A \in M_{n\times n}(\Re)$ and let U be the subset of $M_{n\times n}(\Re)$ defined by

$$U = \{X \in M_{n\times n}(\Re) \mid AX = XA\}$$

(a) Prove that U is a subspace of $M_{n\times n}(\Re)$.

(b) Let V be the set of matrices of the form

$$a_0 I_n + a_1 A + \cdots + a_m A^m, \; a_0, a_1, \ldots, a_m \in \Re.$$

Prove that V is a subspace of $M_{n\times n}(\Re)$ and that $V \subset U$.

(c) Find spanning families for U and V when

- $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
- $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad \lambda \neq \mu$

(Hint: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^2 = (a+d)A - (ad - bc)I_2$.)

10. Which of the following sets of vectors in $\Re^3$ are linearly independent?

   (a) (2,-1,2),(3,0,1),(2,2,2)

   (b) (3,1,1),(2,-1,5),(1,7,-17)

   (c) (6,0,-1),(1,1,4)

   (d) (1,3,3),(0,1,4),(5,6,3),(7,2,-1)

11. Which of the following sets of vectors in P_2 are linearly independent?

   (a) $2 - x + 4x^2,\ 3 + 6x + 2x^2,\ 2 + 10x - 4x^2$

   (b) $3 + x + x^2,\ 2 - x + 5x^2,\ 4 - 3x^2$

   (c) $6 - x^2,\ 1 + x + 4x^2$

   (d) $1 + 3x + 3x^2,\ x + 4x^2,\ 5 + 6x + 3x^2,\ 7 + 2x - x^2$

12. Let α and β be distinct real numbers. Prove that the vectors $(1, \alpha)$ and $(1, \beta)$ are linearly independent.

13. Let α, β and γ be distinct real numbers. Prove that the vectors $(1, \alpha, \alpha^2)$, $(1, \beta, \beta^2)$ and $(1, \gamma, \gamma^2)$ are linearly independent.

14. Let $u_1, u_2, \ldots, u_n$ be a linearly independent family of vectors in V and let vectors $v_1, v_2, \ldots, v_m \in V$ be defined by

$$v_i = \sum_{j=1}^{n} a_{ij} u_j, \quad 1 \leq i \leq m$$

   Prove that $v_1, v_2, \ldots, v_m$ are linearly independent if and only if the rows of the matrix $A = [a_{ij}]$ are linearly independent.

15. Prove that three vectors in $\Re^n$ are linearly dependent if and only if they lie in a plane.

Tutorial Sheet Three

1. Explain why the following sets of vectors are *not* bases for the indicated vector spaces. (Solve this problem by inspection).

   (a) $\mathbf{u_1} = (1,2), \mathbf{u_2} = (0,3), \mathbf{u_1} = (2,7)$ for $\Re^2$

   (b) $\mathbf{u_1} = (1,2,1), \mathbf{u_2} = (0,3,2)$ for $\Re^3$

   (c) $\mathbf{p_1} = 1 + x + x^2, \mathbf{p_2} = x - 1$ for P_2

2. Which of the following sets of vectors are bases for $\Re^3$?

   (a) (1,0,0),(2,2,0),(3,3,3)

   (b) (3,1,-4),(2,5,6),(1,4,8)

   (c) (2,-3,1),(4,1,1),(0,-7,-1)

   (d) (1,6,4),(2,4,-1),(-1,2,5)

3. Which of the following sets of vectors are bases for P_2?

   (a) $1 - 3x + 2x^2,\ 1 + x + 4x^2,\ 1 - 7x$

   (b) $4 + 6x + x^2,\ -1 + 4x + 2x^2,\ 5 + 2x - x^2$

   (c) $1 + x + x^2,\ x + x^2,\ x^2$

   (d) $-4 + x + 3x^2,\ 6 + 5x + 2x^2,\ 8 + 4x + x^2$

4. Show that the following set of vectors is a basis for $M_{2\times2}(\Re)$.

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

In Questions 5 and 6 determine the dimension and a basis for the solution space of the system

5. $$\begin{aligned} 3x_1 + x_2 + 2x_3 &= 0 \\ 4x_1 \qquad\quad + 5x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

7. Determine bases for the following subspaces of $\Re^3$.

   (a) The plane $3x - 2y + 5z = 0$

   (b) The plane $x - y = 0$

   (c) The line described by the parametric equations

$$\begin{aligned} x &= t \\ y &= -t \qquad -\infty < t < \infty \\ z &= 4t \end{aligned}$$

8. Determine the dimensions of the following subspaces of $\Re^4$.

   (a) All vectors of the form $(a, b, c, 0)$

   (b) All vectors of the form (a, b, c, d) where $d = a + b$ and $c = a - b$

   (c) All vectors of the form (a, b, c, d) where $a = b = c = d$

9. Determine the dimension of the subspace of P^3 consisting of all polynomials $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ for which $a_0 = 0$.

10. Let $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis for a vector space V. Show that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is also a basis, where $\mathbf{u_1} = \mathbf{v_1}$, $\mathbf{u_2} = \mathbf{v_1} + \mathbf{v_2}$ and $\mathbf{u_3} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}$.

11. Show that the vector space of all real-valued functions defined on the entire real line is infinite dimensional. (**Hint:** Assume it is finite dimensional with dimension n, and obtain a contradiction by producing $n+1$ linearly independent vectors.)

12. Let V be a subspace of a finite dimensional vector space W. Show that $\dim(V) \leq \dim(W)$.

13. Show that the only subspaces of $\Re^3$ are lines through the origin, planes through the origin, the zero subspace, and $\Re^3$ itself. (**Hint:** It is known that the subspaces of $\Re^3$ must be 0-dimensional, 1-dimensional, 2-dimensional or 3-dimensional.)

14. Find the coordinate vector of $\mathbf{v} = (7,4)^T$ relative to the basis $(3,2)^T, (1,1)^T$ of $\Re^2$.

15. $\mathbf{v_1} = (1,1,1)^T, \mathbf{v_2} = (2,3,2)^T, \mathbf{v_3} = (1,5,4)^T$ form a basis β for $\Re^3$. Vectors $\mathbf{u_1} = (1,1,0)^T, \mathbf{u_2} = (1,2,0)^T, \mathbf{u_3} = (1,2,1)^T$ form a basis γ for $\Re^3$. Find the change of basis matrix $[P]_\beta^\gamma$. Use this matrix to find $[3\mathbf{v_1} + 2\mathbf{v_2} - \mathbf{v_3}]_\gamma$.

16. Let V be the vector space $\Re^2$. Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1+x, 1+x^2, x+x^2\}$ be two bases for V. Caluculate the change of basis matrix $[P]_\beta^\gamma$. Use this to calculate $[3 + x + -2x^2]_\gamma$.

17. Let β be a basis for an n-dimensional vector space V. Show that if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ form a linearly independent set of vectors in V, then the coordinate vectors $[\mathbf{v_1}]_\beta, [\mathbf{v_2}]_\beta, \ldots, [\mathbf{v_r}]_\beta$ form a linearly independent set in $\Re^n$. Similarly show that if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ span V, then the coordinate vectors $[\mathbf{v_1}]_\beta, [\mathbf{v_2}]_\beta, \ldots, [\mathbf{v_r}]_\beta$ span $\Re^n$. Conclude that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ forms a basis for V if and only if $[\mathbf{v_1}]_\beta, [\mathbf{v_2}]_\beta, \ldots, [\mathbf{v_n}]_\beta$ form a basis for $\Re^n$.

18. Find an orthonormal basis for the subspace of $\Re^4$ spanned by

$$\mathbf{u_1} = (1,1,1,1)^T, \ \mathbf{u_2} = (0,1,1,1)^T, \ \mathbf{u_3} = (0,0,1,1)^T$$

Extend this to an orthonormal basis for $\Re^4$.

Tutorial Sheet Four

1. Find bases for the row space, the column space and the null space of the following matrices. Verify for each matrix that $\dim R(A) = \dim C(A)$ and that $\text{rank}(A) + \text{nullity}(A) = n$.

$$\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 0 & 0 & -8 \end{bmatrix} \quad \begin{bmatrix} -3 & 5 & 1 & 2 \\ 7 & 2 & 0 & -4 \\ -8 & 3 & 1 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -2 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -3 & 6 & 6 & 3 \\ 5 & -3 & 10 & 10 & 5 \end{bmatrix}$$

2. Find a basis for the subspace of $\Re^4$ spanned by the given vectors.
   (a) (1,1,-4,-3),(2,0,2,-2),(2,-1,3,2)
   (b) (-1,1,-2,0),(3,3,6,0),(9,0,0,3)
   (c) (1,1,0,0),(0,0,1,1),(-2,0,2,2),(0,-3,0,3)

3. Find a basis for the subspace of P_2 spanned by the given vectors.
   (a) $-1 + x - 2x^2,\ 3 + 3x + 6x^2,\ 9$
   (b) $1 + x,\ x^2,\ -2 + 2x^2,\ -3x$
   (c) $1 + x - 3x^2,\ 2 + 2x - 6x^2,\ 3 + 3x - 9x^2$

4. Find a basis for the subspace of $M_{2\times 2}(\Re)$ spanned by the given vectors.
   (a) $\begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$
   (b) $\begin{bmatrix} -1 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -3 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix}$

5. Show that the nonzero row vectors in any row-echelon form of a matrix A form a basis for the row space of A.

6. Show that row vectors in an $n \times n$ invertivle matrix A form a basis for $\Re^n$.

7. U and V are subspaces of $\Re^5$ where, $U = \text{span}(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$, $V = \text{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$, where $u_1, u_2, u_3, v_1, v_2, v_3$ are the respective columns of the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 & 1 \\ 3 & 4 & 9 & 6 & 8 & 3 \\ -3 & -1 & 0 & 2 & -1 & -1 \\ -1 & -2 & -5 & -2 & -6 & -5 \\ -4 & -2 & -2 & 3 & -5 & -6 \end{bmatrix}$$

(a) Assuming that A has reduced row-echelon form

$$B = \begin{bmatrix} 1 & 0 & -1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 0 & 5 & 6 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

find bases for each of the subspaces $U, V, U + V$.

8. (a) If $a_1, a_2, \ldots, a_n$ are not all zero, prove that the set

$$U = \{(x_1, x_2, \ldots, x_n)^T \mid a_1x_1 + \cdots + a_nx_n = 0\} \tag{1}$$

is a subspace of $\Re^n$ with dimension equal to $n - 1$.

(b) Prove conversely that any supspace U of $\Re^n$ having dimension equal to $n - 1$ must have the form (1).

9. A is a 3×3 matrix such that $A^2 = 0$ and $A \neq 0$. Prove that

(a) $C(A) \subset N(A)$. (Hint: Let $X \in C(A)$. Then $X = AY$ for some $Y \in \Re^3$.)

(b) $\text{rank}(A) = 1$. (Hint: Use the *rank* + *nullity* theorem.)

(c) Exhibit a nonzero 3×3 matrix A such that $A^2 = 0$.

10. Suppose that $A \in M_{n\times n}(\Re)$ satisfies $A^2 = A$. Prove the following

(a) If $Y \in C(A)$, then $AY = Y$.

(b) $N(A)\hat{C}(A) = \{\mathbf{0}\}$.

(c) $\Re^n = N(A) + C(A)$.

Tutorial Sheet Five

1. $T : \Re^2 \to \Re^2$ is a linear transformation which maps $(1,2)^T$ to $(-2,3)^T$ and $(1,-1)^T$ to $(5,2)^T$. Find $T(\mathbf{v})$ when $\mathbf{v} = (7,5)^T$.

2. Let $T : U \to V$ be a linear transformation. If $\text{Ker}T = \{\mathbf{0}\}$ and $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ are linearly independent in U, prove that $T(\mathbf{u_1}), \ldots, T(\mathbf{u_2})$ are linearly independent in V.

3. Let $T : U \to V$ be a linear transformation. Prove that T is 1-1 (ie. $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$) if and only if $\text{Ker}T = \{\mathbf{0}\}$.

4. Let $T : U \to V$ be a linear transformation. Using the $rank + nullity$ theorem, prove that

   (a) $\text{rank}(T) \leq \dim(U)$

   (b) $\text{Ker}(T) = \{\mathbf{0}\} \Rightarrow \dim(U) \leq \dim(V)$

5. Let U be a vector space with basis $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$. $T : U \to U$ is the linear transformation defined by

$$\begin{aligned} T(\mathbf{u_1}) &= \mathbf{u_1} + \mathbf{u_2} + \mathbf{u_3} \\ T(\mathbf{u_2}) &= \mathbf{u_1} - \mathbf{u_2} + \mathbf{u_3} \\ T(\mathbf{u_3}) &= 2\mathbf{u_1} + 2\mathbf{u_3} \end{aligned}$$

   Find bases for $\text{Ker}(T)$, $\text{Im}(T)$. Also find $\text{rank}(T)$ and $\text{nullity}(T)$.

6. Let U be a vector space with basis $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$. $T : U \to U$ is the linear transformation defined by

$$T(\mathbf{u_1}) = \mathbf{u_3} \quad T(\mathbf{u_2}) = -\mathbf{u_3} \quad T(\mathbf{u_3} = \mathbf{u_1} + \mathbf{u_2}$$

   Find bases for $\text{Ker}(T)$, $\text{Im}(T)$. Also find $\text{rank}(T)$ and $\text{nullity}(T)$.

7. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and let $T : M_{2\times 2}(\Re) \to M_{2\times 2}(\Re)$ be the linear transformation defined by $T(X) = AX - XA$. Prove that

$$\mathrm{Im}(T) = \mathrm{span}\left\{\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}\right\}, \ \mathrm{Ker}T = \mathrm{span}\{I_2, A\}$$

Write down $\mathrm{rank}(T)$ and $\mathrm{nullity}(T)$.

8. If $T : U \to V$ is a linear transformation, prove that $\mathrm{Im}(T)$ is a subspace of V.

9. Let $T : U \to V$ be a linear transformation, where $\dim(U) = \dim(V)$. If $\mathrm{Ker}(T) = \{\mathbf{0}\}$, deduce that $\mathrm{Im}(T) = V$.

10. Let $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ be a basis for $\Re^3$ and $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be any vectors in $\Re^3$. If T is a linear transformation such that $T(\mathbf{u_i}) = \mathbf{v_i}$ for $i = 1, 2, 3$, show that $T = T_A$, where A is the 3×3 matrix

$$A = \left[\ \mathbf{v_1} \mid \mathbf{v_2} \mid \mathbf{v_3}\ \right]\left[\ \mathbf{u_1} \mid \mathbf{u_2} \mid \mathbf{u_3}\ \right]^{-1}$$

11. Suppose $V = M_{2\times 2}(\Re)$ and $\beta : E_{11}, E_{12}, E_{21}, E_{22}$ is the standard basis for V. Mappings $S, T : V \to V$ are defined by

$$T(A) = \frac{1}{2}(A - A^T), \ S(A) = \frac{1}{2}(A + A^T)$$

(a) Prove that S and T are linear.

(b) Find $[S]_\beta^\beta$ and $[T]_\beta^\beta$.

(c) Find bases for $\mathrm{Ker}(S)$ and $\mathrm{Im}(S)$, $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

(d) Prove that $S^2 = S, T^2 = T$.

(e) Prove that $ST = 0, TS = 0$

(f) Prove that $S + T = I_V$, where $S + T$ is the linear mapping defined by $(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$.

12. Let γ : $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ be the standard basis of unit vectors for $V = \Re^3$ and let $\beta : \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be the basis of $\Re^3$ given by

$$\mathbf{v_1} = [1, 1, -1]^T, \ \mathbf{v_2} = [2, 1, 3]^T, \ \mathbf{v_3} = [0, 1, 1]^T$$

Find $[I_V]^\gamma_\beta$ and $[I_V]^\beta_\gamma$.

13. Let A and B be non-singular $n \times n$ matrices over $\Re$ and let $V = M_{n\times n}(\Re)$. Let $X \in V$. Show that the mapping $T : V \to V$ defined by $T(X) = AXB$ has the property that $\text{Ker}(T) = \{\mathbf{0}\}$ and $\text{Im}(T) = V$.

14. A mapping $T : P_2[\Re] \to \Re^3$ is defined by

$$T(f(x)) = \begin{bmatrix} f(1) \\ f(0) \\ f(-1) \end{bmatrix}$$

(a) Prove that T is a linear transformation.

(b) If $S : \Re^3 \to P_2[\Re]$ is the linear transformation defined by

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = b + \frac{a-c}{2}x + \frac{a-2b+c)}{2}x^2$$

verify that $ST = I_{P_2[\Re]}$ and $TS = I_{\Re^3}$.

15. Let $T : P_2[\Re] \to P_2[\Re]$ be given by $T(f(x)) = f'(x)g(x) + 2f(x)$, where $g(x) = 3 + x$ and $f'(x)$ is the formal derivative of f (ie. if $f = a_0 + a_1x + a_2x^2$, then $f'(x) = a_1 + 2a_2x$, where $a_0, a_1, a_2 \in \Re$).

Also let $S : P_2[\Re] \to \Re^3$ be defined by $S(a + bx + cx^2) = [a + b, c, a - b]^T$, where $a, b, c \in \Re$.

Let $\beta : 1, x, x^2$ and $\gamma : \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ be the standard bases for $P_2[\Re]$ and $\Re^3$ respectiely.

Find $[S]^\gamma_\beta, [T]^\beta_\beta$ and $[ST]^\gamma_\beta$ and verify that $[ST]^\gamma_\beta = [S]^\gamma_\beta[T]^\beta_\beta$.

16. Let $T : P_4[\Re] \to P_4[\Re]$ be the linear transformation defined by

$$T(f(x)) = \frac{1}{2}(f(x) + f(-x)).$$

(a) Prove that $T^2 = T$.

(b) For the basis $\beta : 1, x^2, x^4, x, x^3$ of $P_4[\Re]$, find $[T]_\beta^\beta$.

In exercises 17 and 18 find $[T]_\beta^\beta$ and use Theorem 6.12 to calculate $[T]_\gamma^\gamma$.

17. Let $\beta = \{\mathbf{u_1}, \mathbf{u_2}\}$ and $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$ be two bases for $\Re^2$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$T : \Re^2 \to \Re^2$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_2 \end{bmatrix}$$

18. Let $\beta = \{\mathbf{u_1}, \mathbf{u_2}\}$ and $\gamma = \{\mathbf{v_1}, \mathbf{v_2}\}$ be two bases for $\Re^2$, where

$$\mathbf{u_1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \mathbf{u_2} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$T : \Re^2 \to \Re^2$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 7x_2 \\ 3x_1 - 4x_2 \end{bmatrix}$$

19. Prove that similar matrices, have the same rank and nullity.

20. Prove that if A and B are similar matrices, then A^k and B^k are also similar, where k is any positive integer.

Tutorial Sheet Six

1. If λ is an eigenvalue of A, prove that λ^2 is an eigenvalue of A^2.

2. Let $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

   (a) Verify that $ch_A(x) = (x-1)(x+\frac{1}{2})^2$

   (b) Find $g_A(1)$ and $g_A(-\frac{1}{2})$ and deduce that A is diagonalizable.

   (c) Find a non-singular matrix $P \in M_{n\times n}(\Re)$ such that $P^{-1}AP$ is diagonal.

3. For each of the following matrices, determine if they are diagonalizable. If so, find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & -6 \\ 3 & 5 & 8 \\ -6 & 8 & 4 \end{bmatrix} \quad \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 2 & 3 \end{bmatrix}$$

4. Evaluate A^m for each matrix A. Use diagonalization to shorten the process.

   (a) $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

5. Solve the following system of linear differential equations by using an appropriate change of variables.

$$\begin{aligned} x_1' &= -x_1 + 2x_2 + 3x_3 \\ x_2' &= 2x_1 + -3x_2 + 4x_3 \\ x_3' &= 3x_1 + 4x_2 + -2x_3 \end{aligned}$$

6. Let

$$A = \begin{bmatrix} 1 & -4 & 0 \\ -4 & 3 & -4 \\ 0 & -4 & 5 \end{bmatrix}$$

Find an orthogonal matrix P such that $P^TAP = D$ where D is diagonal.

7. Let

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & -2 \\ -2 & -2 & 5 \end{bmatrix}$$

Find an orthogonal matrix P such that $P^TAP = D$ where D is diagonal.

8. If A is a real symmetric matrix and λ and μ are distinct eigenvalues of A, with corresponding eigenvectors X and Y, prove that $X^TY = 0$.

9. If A is a real symmetric matrix, prove that $N(A^2) = N(A)$. (Hint: Assume $A^2X = 0$. Then $A^TAX = 0$ etc.)