MT152

1. Functions of Several Variables.

1.1 Parabolas, Circles, Elipses & Hyperbolas.

Linear Functions

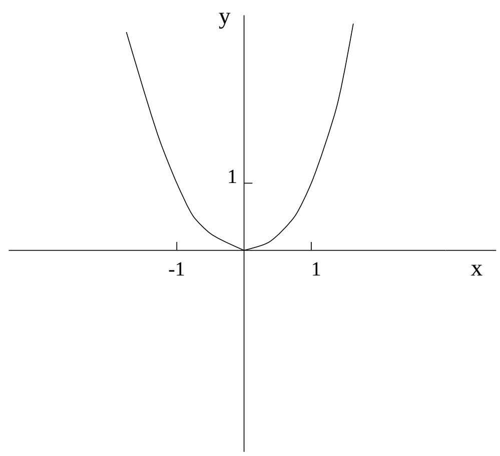
A line $y = mx + c$ has slope m and y intercept c.

Or $(y - y_0) = m(x - x_0)$ is a line slope m passing through (x_0, y_0).

Parabolas $y = f(x) = ax^2 + bx + c$ ($f(x)$ is a quadratic function of x.)

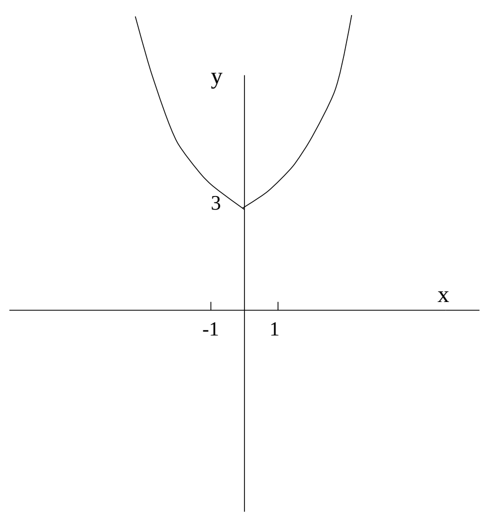
Say

$y = x^2$



Or

$y = x^2 + 3$



which is the same curve shifted up 3

Or if

$$y = (x + 2)^2$$

the curve is shifted to the left by 2 and if $y = 2x^2$ it is stretched.

To sketch a parabola $y = x^2/2 - 3x + 4$, complete the square in x;

$$y = (x^2 - 6x)/2 + 4$$

$$y = ((x - 3)^2 - 9)/2 + 4$$

$$y = ((x - 3)^2 - 1)/2.$$

As $x \to \pm\infty \quad y \to \infty$, ie. parabola points up, with lowest point at $x = 3$ $\Rightarrow y = -1/2$, ie. at $(3, -\frac{1}{2})$.

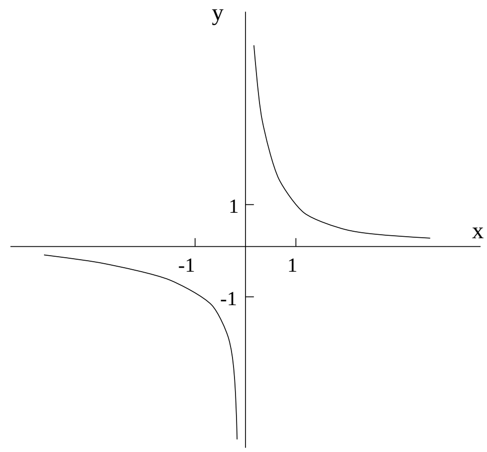
Example: Sketch $y = -3x^2 - 6x + 1$

Parabolas, Circles, Ellipses and Hyperbolas are all examples of conic sections. That means they all come from intersecting a double cone (or useless egg timer) such as $z^2 = x^2 + y^2$ with a plane. Slice horizontally and you get circles. Tip the plane a bit and you get ellipses. Keep tipping (so slope is > 1) and you will get a parabola. Finally if the plane is vertical the intersection is a hyperbola.

Hyperbolas

The simplest example is $y = 1/x$.

$$y = \frac{1}{x}$$



The x and y axes are asymptotes, because as $x \to \infty y \to 0$ (x axis)

and as $y \to \infty x \to 0$ (y axis).

If the equation is written as $xy = 1$ you can see that it is symmetrical in x and y.

Now consider $x^2 - y^2 = 1$.

For x and y large $x^2 - y^2 = 0$ so that

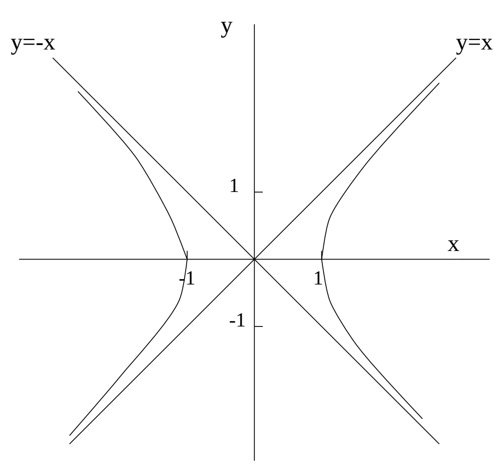
$$x = y \text{ or } x = -y.$$

So $x = y$ and $x = -y$ are asymptotes.

Now if $y = 0$ then $x = 1$ or -1. So the hyperbola passes through $(1, 0)$ and $(-1, 0)$.

(Recall the egg timer slice a cone $z^2 = x^2 + y^2$ vertically $y = y_0$ and you get the hyperbola $z^2 - x^2 = y_0^2$.)

$$x^2 - y^2 = 1$$



Example: Sketch the hyperbola $x^2 - 4y^2 = 9$

Example: Sketch the hyperbola $x^2 - 2xy = 3$.

Circles. A circle with radius r whose center is at origin has the equation $x^2 + y^2 = r^2$. Which can be represented parametrically through

$$x = r\cos\theta, \quad y = r\sin\theta \qquad (\text{because } \sin^2\theta + \cos^2\theta = 1)$$

For a circle centre $(2, -3)$, with radius 2 we have $(x-2)^2 + (y+3)^2 = 4$, which can also be represented parametrically:

$$(x-2) = 2\cos\theta, \quad (y+3) = 2\sin\theta \rightarrow x = 2 + 2\cos\theta, \quad y = -3 + 2\sin\theta.$$

Example: Sketch the circle $x^2 + 10x + y^2 - 4y + 20 = 0$

Ellipses are just squashed circles. Centred at the origin and squashed parallel to the x or y axes they have the form

$$(x/a)^2 + (y/b)^2 = 1.$$

(Parametric representation $x = a\cos\theta$, $y = b\sin\theta$.)

Example: Sketch the ellipse $x^2 - x + 9y^2 = 0$.