1. Functions of Several Variables.

1.2 Points and planes in 3 space (distance).

Usually 3 space $(x,\, y,\, z)$ has z pointing up wards. Then x and y give the position on the ground and z the height.

Horizontal and Vertical Planes

The x and y axes lie in the **horizontal plane** $z = 0$. All other **horizontal planes** are parallel to $z = 0$ and are given by $z =$ constant.

Vertical planes, for instance, $x = 3$ or $y = -1$ have the equation of straight lines in x and y (z must not appear) – that is $ax + by + d = 0$, or a linear function of x and y equals zero.

Example: Sketch $\quad x + y - 1 = 0.$

First sketch the line in the $(x,\, y)$ plane, then the plane contains this line and is parallel to the z axis.

Planes in General

For a general plane the linear function depends on x, y and z

$$\text{ie.} \quad f(x,\, y,\, z) = ax + by + cz + d = 0$$

which can be rearranged into the form

$$z = \bar{f}(x,\, y) = \bar{a}x + \bar{b}y + \bar{d},$$

provided $c \neq 0$.

Example: Sketch the plane

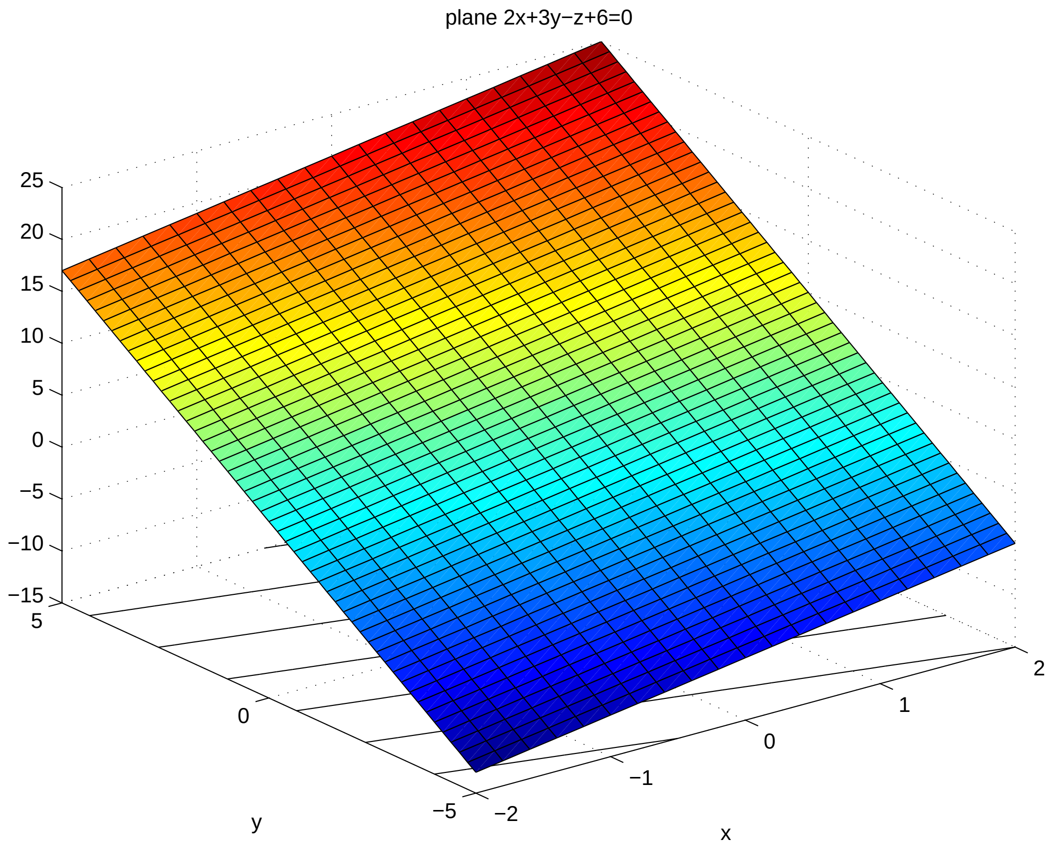
$$2x + 3y - z + 6 = 0.$$

First find the intercepts on the axes

$$
\begin{array}{llll}
x \text{ axis} & (y=0,\ z=0) & \Rightarrow & x=-3 \\
y \text{ axis} & (x=0,\ z=0) & \Rightarrow & y=-2 \\
z \text{ axis} & (x=0,\ y=0) & \Rightarrow & z=6.
\end{array}
$$

Draw the triangle between these 3 points and try to imagine the plane!

The triangle defines the plane.



In fact if you know any two lines in a plane you know the plane.

Example: Find the plane which intersects the xz plane in the line $z = 3x + 2$ and the yz plane in the line $z = 2 - 2y$.

Let $z = \bar{a}x + \bar{b}y + \bar{d}$.

Then if

$$
\begin{array}{rllll}
\text{when} & y = 0 & (xz \text{ plane}) & \text{then } z = 3x + 2 \ \Rightarrow & \bar{a} = 3,\ \bar{d} = 2 \\
\text{and when} & x = 0 & (xz \text{ plane}) & \text{then } z = 2 - 2y & \bar{b} = -2 \quad " \\
 & & \text{so that} & \Rightarrow\ z = 3x - 2y + 2 &
\end{array}
$$

Example: What is the plane passing through

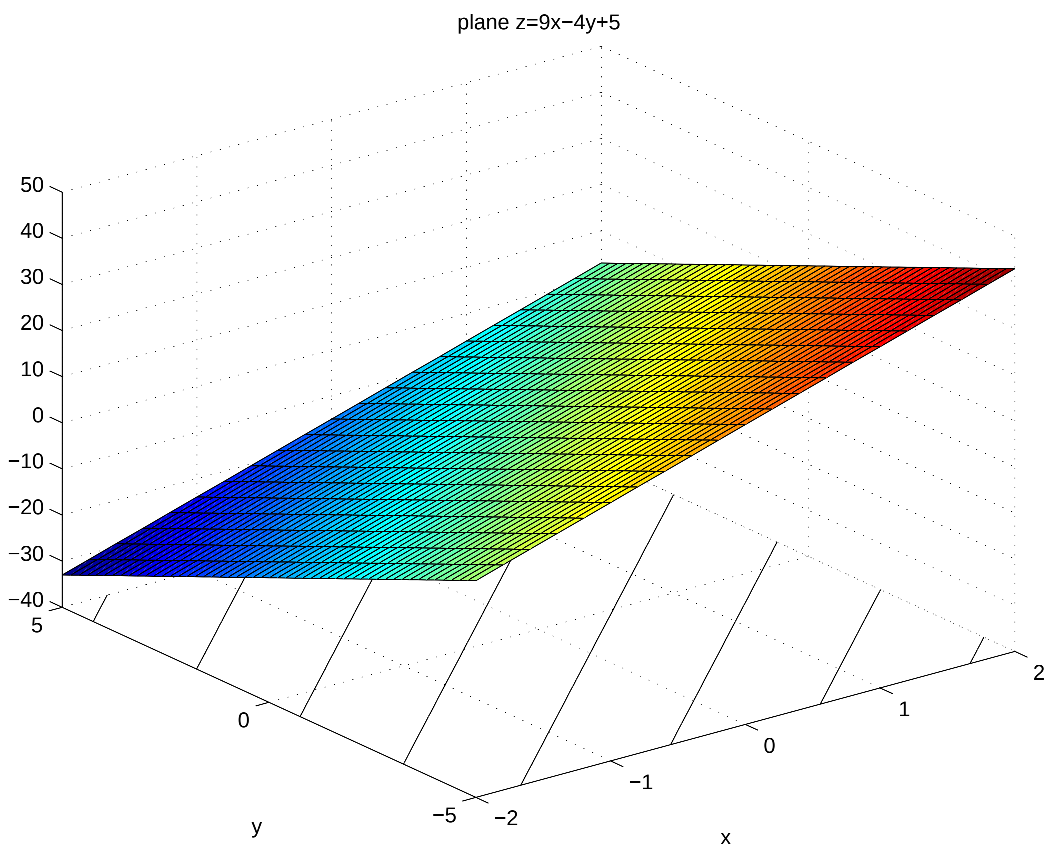
$$(0,\,0,\,5)\,,\quad (1,\,3,\,2)\,,\quad (0,\,1,\,1).$$

Since $x = mx + ny + c$ the first point gives $c = 5$.

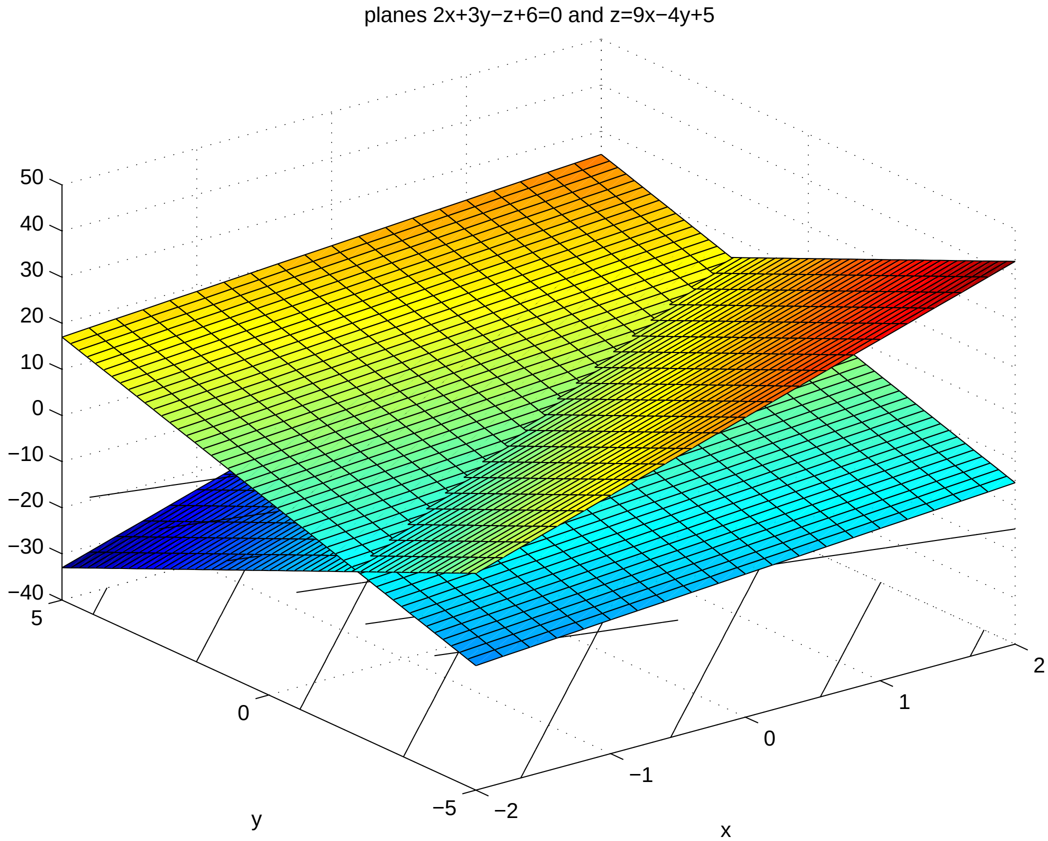
The 2nd point gives $2 = m + 3n + 5$

and the 3rd point gives $1 = n + 5$.

Solving simultaneously gives $m = 9$ and $n = -4$. So the plane is $z = 9x - 4y + 5$.



The following pictures shows the intersection of the planes $2x+3y-z+6=0$ and $z=9x-4y+5$. Note that the intersection is a line.



Distance in 3d

In 2d Pythagoras gives the distance between (x_0, y_0) and (x_1, y_1) as

$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

It is not hard to show that in 3d

$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

Example: Find distance between the points $(1, 2, 3)$ and $(0, 0, 0)$.

$$d = \sqrt{1 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}.$$

Example: Find the distance from the point (1, 2, 3) to the plane $z = 4$.

The distance intended here is the shortest distance – or the distance perpendicular to the plane. In this case the direction perpendicular to $z = 4$ is the z direction. So the distance is $4 - 3 = 1$. However say the plane were $x + y = 1$. Since it is parallel to the z axis we can work in 2d; (x, y). From the graph we see that the distance is $\sqrt{2}$.

(See later for a vector representation of a plane.)