MT152

2. Partial Derivative and Tangent Planes

2.1 Partial Derivatives and Surfaces

Take the surface $z = f(x, y) = 1 - x^2 - y^2$.

Consider standing at $(1, 0, 0)$ and walking in the x direction. Then you would be walking in the plane $y = 0$ along the curve

$$z = f(x, 0) = 1 - x^2.$$

The slope of this curve is simply the rate of change of $f(x, 0)$ with respect to x. ($= -2x$ here). Now let's be more general and stand at (a, b) on $z = f(x, y) = 1-x^2-y^2$ and walk in the x direction. You still walk along a curve which only changes in x; $z = 1 - x^2 - b^2$. You are walking in the plane $y = b$. (b constant).

This slope is called the *Partial Derivative* of $f(x, y)$ with respect to x. Formally

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}.$$

Sometimes $\frac{\partial f}{\partial x}(a, b)$ is denoted $f_x(a, b)$.

Similarly

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

is the rate of change of f with respect to y.

To calculate, say $\frac{\partial f}{\partial x}(a, b)$, treat y as a constant, differentiate, then evaluate at (a, b)

eg. if $f(x, y) = ye^{2x}$

$$\frac{\partial f}{\partial x} = y\frac{de^{2x}}{dx} = 2ye^{2x}.$$

Here y is a constant that multiplies the function of x being differentiated. But $\frac{\partial f}{\partial x}$ is still a function of x and y.

So

$$\frac{\partial f}{\partial x}(1, 3) = 3.2e^2 = 6e^2.$$

Now

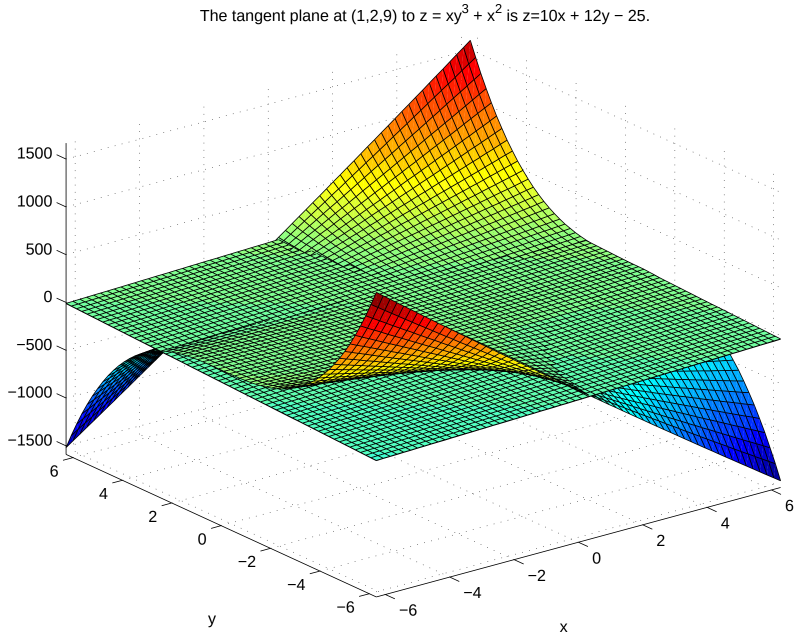
$$\frac{\partial f}{\partial y} = e^{2x} \cdot \frac{d(y)}{dy} = e^{2x}$$

so

$$\frac{\partial f}{\partial y}(1,\, 3) = e^2.$$

Example: Find $f_x(1,\, 2)$ and $f_y(1,\, 2)$ if

$$f(x,\, y) = xy^3 + x^2.$$



Example: Find $f_x(0,\, \frac{\pi}{2})$ and $f_y(0,\, \frac{\pi}{2})$ if $f(x,\, y) = \sin(x+y)e^{2x}$

$$\begin{aligned} f_x(x,\, y) &= \cos(x+y)e^{2x} + 2e^{2x}\sin(x+y) \quad \text{(product rule)} \\ \Rightarrow \quad f_x(0,\, \frac{\pi}{2}) &= \cos(\frac{\pi}{2})e^0 + 2e^0 \sin\frac{\pi}{2} = 2 \\ \text{and} \quad f_y(x,\, y) &= \cos(x+y)e^{2x} \\ \Rightarrow \quad f_y(0,\, \frac{\pi}{2}) &= 0 \end{aligned}$$

Partial Derivatives can also be found for functions of 3 or more variables.

For example the volume of a box

$$V(a,\, b,\, c) = abc.$$

Suppose we let a change by a small amount Δa. What is the change in V?

Graphically we can see that $\Delta V = bc\Delta a \Rightarrow \dfrac{\Delta V}{\Delta a} = bc \underset{\text{let } \Delta a \to 0}{\Rightarrow} \dfrac{\partial V}{\partial a} = bc$ which agrees with the partial differentiation.

Example. On the surface of earth the force of gravity is approximately constant $(-g)$. But out in space it is given by

$F = \frac{GMm}{r^2}$ G is gravitational constant

M is mass of earth

m is mass of rocket

r is distance between centre of mass of the earth and the centre of mass of the rocket.

What is the rate of change of F with respect to r?

$$\frac{\partial F}{\partial r} = GMm\left(\frac{-2}{r^3}\right) = -\frac{2GMm}{r^3} \quad \text{always decreasing.}$$

Can we always find partial derivatives?

In the previous example $\dfrac{\partial F}{\partial r}$ is undefined at $r = 0$, but then so is F.

In fact things can be even worse. Take this function

$$f(x,\, y) = \frac{x^2}{x^2 + y^2},$$

it is not even continuous, let alone differentiable at $(0,\, 0)$.

Definition. If $f(x,\, y)$ is continuous at $(a,\, b)$

$$\lim_{(x,y)\to(a,b)} f(x,\, y)$$

must exist and give the same value which ever way you approach $(a,\, b)$.

For $f(x, y) = \dfrac{x^2}{x^2 + y^2}$

if we approach along $y = 0$ $\lim_{(x,y)\to(0,0)} = 1$,

if we approach along $x = 0$ $\lim_{(x,y)\to(0,0)} = 0$.

So $f(x, y)$ is not continuous at $(0, 0)$ and therefore has no partial derivatives at $(0, 0)$.

However most of the functions we will use are continuous and do have partial derivatives.

Then in some sense the surface is "smooth" and we can define tangent planes.