MT152
2. Partial Derivative and Tangent Planes

2.2 The Tangent Plane and Linear Approximations

Local Linearity

If a surface $z = f(x, y)$ is smooth at (a, b)

$$\text{ie. } f(x, y) \text{ is continuous at } (a, b)$$
$$\text{and } f_x \text{ and } f_y \text{ are defined at } (a, b)$$

then when you "zoom in" close enough the surface will "look like a plane". One way to see this is to look at the contours. The contours of a plane are straight parallel lines the same perpendicular distance apart. So as you zoom into a smooth surface the contours straighten out. This means that close to (a, b) the surface is approximated by a plane - in fact it can be approximated by the tangent plane.

The Tangent Plane.

Consider any plane passing through $(a, b, f(a, b))$

$$z - f(a, b) = m(x - a) + n(y - b).$$

If this is the tangent plane to $z = f(x, y)$ at (a, b) then the intersection of it with the plane $y = b$;

$$z - f(a, b) = m(x - a)$$

must be the tangent line to $z = f(x, b)$. Now the tangent line to $z = f(x, b)$ has slope $\dfrac{\partial f}{\partial x}(a, b)$. So $m = \dfrac{\partial f}{\partial x}(a, b)$.
Similarly the intersection of

$$z - f(a, b) = m(x - a) + n(y - b)$$

with the plane $x = a$ is

$$z - f(a, b) = n(y - b),$$

which must be the tangent line to $z = f(a, y)$. So $n = \dfrac{\partial f}{\partial y}(a, b)$.

Finally the *tangent Plane* to $f(x, y)$ at (a, b), is

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

Example. Find the tangent plane to

$$\begin{aligned} z = f(x, y) = 1 - x^2 - y^2 \quad \text{at} \quad (1, 0) \\ \frac{\partial f}{\partial x} = -2x\,, \quad \frac{\partial f}{\partial y} = -2y \\ \frac{\partial f}{\partial x}(1, 0) = -2\,, \quad \frac{\partial f}{\partial y}(1, 0) = 0 \\ \Rightarrow \qquad m = -2, \quad n = 0. \end{aligned}$$

Also $f(1, 0) = 0$, so the tangent plane is

$$\begin{aligned} z &= 0 - 2(x - 1) \\ \Rightarrow \quad z &= -2x + 2. \end{aligned}$$

Example. What is the plane tangent to the surface $z = f(x, y) = 4 - x^2 + 4x - y^2$ at $(1, 1)$?

Completing the square gives

$$\begin{aligned} z = f(x, y) &= 4 - (x - 2)^2 + 4 - y^2 \\ &= -(x - 2)^2 - y^2 + 8 \end{aligned}$$

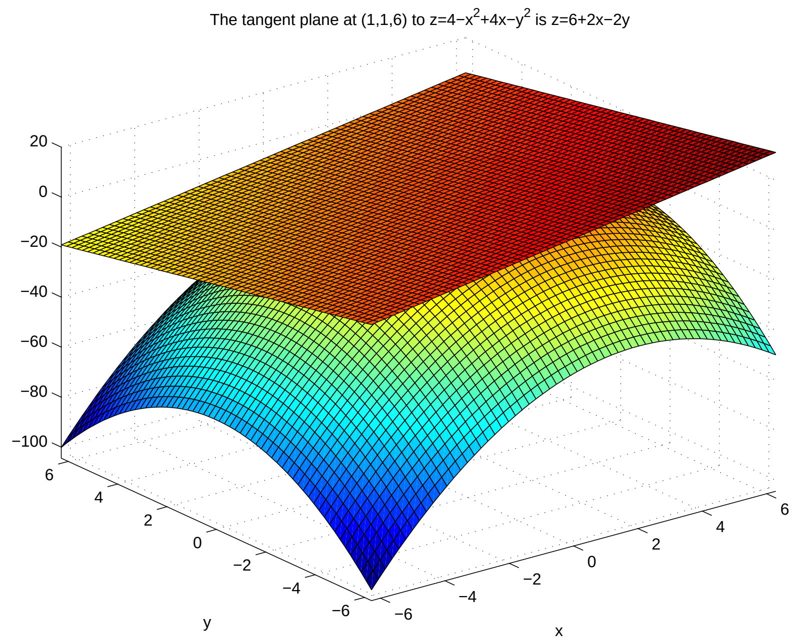
which is a circular paraboloid with axes of symmetry $x = 2$ and $y = 0$, which points down and has maximum at $z = 8$.

To construct the tangent plane at $(1, 1)$ we need $f(1, 1) = -1 - 1 + 8 = 6$, and the partial derivatives at $(1, 1)$.

$$\begin{aligned} \tfrac{\partial f}{\partial x} &= -2(x - 2) && \Rightarrow \tfrac{\partial f}{\partial x}(1, 1) = 2 \\ \tfrac{\partial f}{\partial y} &= -2y && \Rightarrow \tfrac{\partial f}{\partial y}(1, 1) = -2. \end{aligned}$$

So the tangent plane is

$$z = 6 + 2(x - 1) - 2(y - 1).$$



Example. Find the tangent plane to $z = f(x,\, y) = e^{-x^2-y^2}$ at $(1,\, 3)$.

$$f(1,\, 3) = e^{-10}$$
$$\tfrac{\partial f}{\partial x} = -2xe^{-x^2-y^2} \Rightarrow \tfrac{\partial f}{\partial x}(1,\, 3) = -2e^{-10}$$
$$\tfrac{\partial f}{\partial y} = -2ye^{-x^2-y^2} \Rightarrow \tfrac{\partial f}{\partial y}(1,\, e) = -6e^{-10}.$$

The tangent plane at $(1, 3)$ is

$$\begin{aligned} z &= e^{-10} - 2e^{-10}(x-1) - 6e^{-10}(y-3) \\ &= e^{-10}(21 - 2x - 6y). \end{aligned}$$

Linear Approximations.

The tangent plane to $f(x,\, y)$ at $(a,\, b)$,

$$z = f(a,\, b) + f_x(a,\, b)(x - a) + f_y(a,\, b)(y - b)$$

, provides an approximation to the function $z = f(x, y)$ for x close to a and y close to b. So for $(x, y) \approx (a, b)$

$$f(x, y) \simeq f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

provided $f(x, y)$ is smooth near (a, b).

Here the surface is being approximated by a linear function (a plane), so it is often called a *linear approximation* at (a, b).