MT152

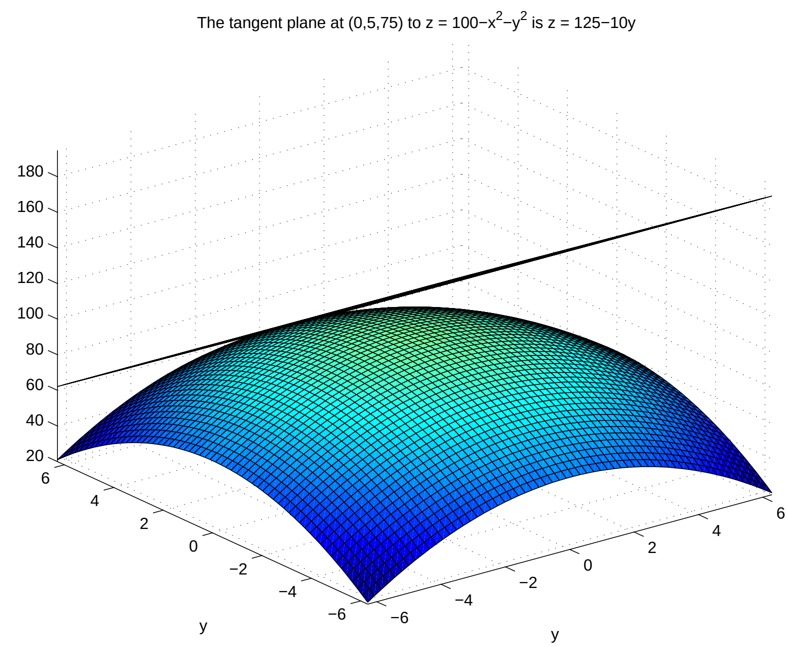
2. Partial Derivative and Tangent Planes

2.3 Applications Using Linear Approximations

Example. Suppose Temperature is given by $T(x, y) = 100 - x^2 - y^2$ in the region $-10 \le x \le 10$, $-10 \le y \le 10$. But we are interested in temperature around $x = 0$ and $y = 5$. Find a linear approximation to $T(x, y)$ about $(0, 5)$.

The linear approximation is the tangent Plane at $(0, 5)$.

$$
\begin{aligned}
&T(0, 5) = 75 \\
&\tfrac{\partial T}{\partial x} = -2x \Rightarrow \tfrac{\partial T}{\partial x}(0, 5) = 0 \\
&\tfrac{\partial T}{\partial y} = -2y \Rightarrow \tfrac{\partial T}{\partial y}(0, 5) = -10 \\
&T(x, y) \simeq 75 - 0(x - 0) - 10(y - 5) \\
&\qquad \simeq 75 - 10(y - 5) = 125 - 10y \qquad \text{for } (x, y) \approx (0, 5).
\end{aligned}
$$



Example. Find the tangent plane to $e^{-x^2} \sin y$ at $(1, \frac{\pi}{2})$ and use it to find an approximate value for $e^{-(0.9)^2} \sin(1.5)$

Let

$$f(x,\, y) = e^{-x^2} \sin y$$
$$f(1,\, \tfrac{\pi}{2}) = e^{-1}.1 = \tfrac{1}{e}$$
$$\tfrac{\partial f}{\partial x} = -2x\, e^{-x^2} \sin y \Rightarrow \tfrac{\partial f}{\partial x}(1,\, \tfrac{\pi}{2}) = -2e^{-1}.1 = \tfrac{-2}{e}$$
$$\tfrac{\partial f}{\partial y} = e^{-x^2} \cos y \Rightarrow \tfrac{\partial f}{\partial y}(1,\, \tfrac{\pi}{2}) = 0.$$

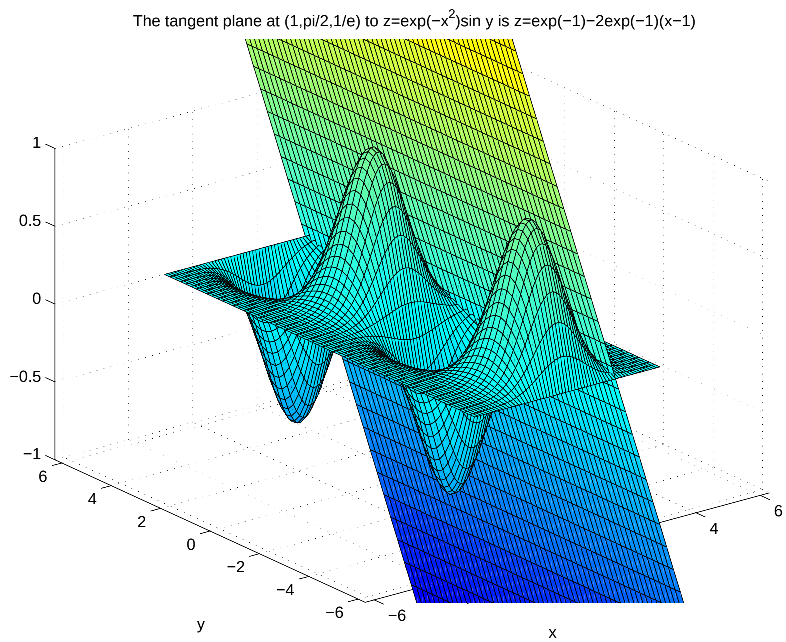
So the tangent plane at $(1,\, \frac{\pi}{2})$ is

$$z = \frac{1}{e} - \frac{2}{e}(x-1).$$

We can use this to approximate $e^{-x^2} \sin y$ for $x \simeq 1$ and $y \simeq \frac{\pi}{2}$.

Here $x = 0.9$ and $y = 1.5$ so

$$\begin{aligned} e^{(0.9)^2} \sin(1.5) &\simeq \frac{1}{e} - \frac{2}{e}(0.9-1) \\ &\simeq \frac{1}{e} + \frac{0.2}{e} = \frac{1.2}{e}. \end{aligned}$$



Example. Electric Power is given by

$$P(E,\, R) = \frac{E^2}{R}$$

where E is voltage and R is resistance.

Find a linear approximation for $P(E, R)$ if

$$E \simeq 200 \text{ volts} \quad \text{and} \quad R \simeq 400 \text{ ohms}.$$

Use this to find the effect that a change in E and R has on P.

$$\frac{\partial P}{\partial E} = \frac{2E}{R} \Rightarrow \frac{\partial P}{\partial E}(200,\, 400) = 1$$
$$\frac{\partial P}{\partial R} = -\frac{E^2}{R^2} \Rightarrow \frac{\partial P}{\partial R}(200,\, 400) = \frac{-1}{4}.$$

So $P(E,\, R) \simeq 100 + (E - 200) + \frac{-1}{4}(R - 400)$ for $(E,\, R) \approx (200,\, 400)$. Let ΔE be the change in E, then $E = 200 + \Delta E$.

Similarly let ΔR be the change in R then $R = 400 + \Delta R$.

So the change in P, $\Delta P = P - 100$ is given approximately by

$$\Delta P \simeq \Delta E + \frac{-1}{4}\Delta R$$

Linear Approximations and small changes

As in the previous example if we let

$$\begin{aligned} \Delta f &= f(x,\, y) - f(a,\, b), \\ \Delta x &= (x - a), \\ \Delta y &= (y - b), \end{aligned}$$

then the linear approximation to $f(x,\, y)$ at $(a,\, b)$, may be written as

$$\Delta f = f_x(a,\, b)\Delta x + f_y(a,\, b)\Delta y.$$

Example. Suppose when making up a metal barrel, base radius 1, height 2, you allow for an error of 5% in radius and height, what is the resulting error in volume.

The volume of the barrel is

$$V(r,\, h) = \pi r^2 h.$$

Now $V_r = 2\pi rh$ and $V_h = \pi r^2$ and at $(r, h) = (1, 2)$,

$$\begin{aligned} V_r(1, 2) &= 4\pi \\ V_h(1, 2) &= \pi. \end{aligned}$$

So $\Delta V = 4\pi\Delta r + \pi\Delta h$.

Now if the error in radius is 5% then

$$\frac{\Delta r}{r} = 0.05 \Rightarrow \Delta r = 0.05, \text{ since } r = 1.$$

If the error in height is 5%

$$\frac{\Delta h}{h} = 0.05 \Rightarrow \Delta h = 0.1, \text{ since } h = 2.$$

So $\Delta V = \pi(4.(0.05) + 0.1) = 0.3\pi$ or the % error in volume

$$\frac{\Delta V}{V} = \frac{0.3\pi}{(1)(2)\pi} = 0.015 \Rightarrow 1.5\%.$$

Infinitesimal changes

In mathematics we often imagine *infinitesimal changes* which are denoted dx rather than Δx, etc. The linear approximation then gives rise to the *differential of the function*

$$df = f_x(a, b)dx + f_y(a, b)dy.$$

which we will use later in section 2.5 on the chain rule.