# MT152
# 2. Partial Derivative and Tangent Planes

## 2.4. Gradients and Directional Derivatives

We can use partial derivatives to work out the slope of $z = f(x, y)$
in the $x$ direction; $\dfrac{\partial f}{\partial x}$, or in the $y$ direction; $\dfrac{\partial f}{\partial y}$.
But what is the slope in the $y = x$ direction?

First we need a more precise definition of the direction using vectors:

Let $\boldsymbol{i}$ be a unit vector in the $x$ direction and let $\boldsymbol{j}$ be a unit vector in the $y$ direction. A *unit* vector is a vector whose magnitude is 1.

A vector in the $y = x$ direction is $\boldsymbol{v} = \boldsymbol{i}+\boldsymbol{j}$. But this is not a unit vector, as it's magnitude $\|\boldsymbol{v}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$. So we divide by $\|\boldsymbol{v}\|$ to get a unit vector.

$$\boldsymbol{u} = \frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j}) \text{ is a unit vector in the } y = x \text{ direction.}$$

Suppose as an example we wanted to find the slope of $f(x, y) = 4 - x^2 - 4y^2$ at $(a, b)$ in the $y = x$ direction. Say we move a short distance $h$ in that direction. Then we could get an approximate slope by calculating the corresponding change in height $f$, that is $\Delta f$. Then the slope is $\simeq \frac{\Delta f}{h}$.

Now since $\boldsymbol{u} = \frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j})$ is a unit vector in the $y = x$ direction it follows that

$$h\boldsymbol{u} = \frac{h}{\sqrt{2}}\boldsymbol{i} + \frac{h}{\sqrt{2}}\boldsymbol{j}$$

is a vector of length $h$ in that direction.

So moving a distance $h$ in the $y = x$ direction implies a distance $\Delta x = \frac{h}{\sqrt{2}}$ in the $x$ direction and a diatance $\Delta y = \frac{h}{\sqrt{2}}$ in the $y$ direction.

Now we can see that the corresponding change in height would be

$$\Delta f = f(a + \frac{h}{\sqrt{2}}, b + \frac{h}{2}) - f(a, b).$$

But $h$ is small so we can use linear approximations;

$$\Delta f \simeq \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

where here $\Delta x = \frac{h}{\sqrt{2}}$ and $\Delta y = \frac{h}{\sqrt{2}}$ and we know how to calculate

$$\frac{\partial f}{\partial x}(a,\, b) \quad \text{and} \quad \frac{\partial f}{\partial y}(a,\, b).$$

So

$$\Delta f \simeq \frac{\partial f}{\partial x}\frac{h}{\sqrt{2}} + \frac{\partial f}{\partial y}\frac{h}{\sqrt{2}},$$

which implies that the slope in the $(\frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{j})$ direction is

$$\frac{\Delta f}{h} \simeq \frac{1}{\sqrt{2}}\frac{\partial f}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial f}{\partial y}.$$

Now let $h \to 0$. Then the rate of change of $f$ is $\frac{1}{\sqrt{2}}\frac{\partial f}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial f}{\partial y}$ in the direction $\boldsymbol{u} = \frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{j}$.

**Back to the example.**

If $f(x,\, y) = 4 - x^2 - 4y^2$ and $(a, b) = (1, 1)$ then

$$\frac{\partial f}{\partial x}(1,\, 1) = -2x\Big|_{(1,1)} = -2$$

and

$$\frac{\partial f}{\partial y}(1,\, 1) = -8y\Big|_{(1,1)} = -8,$$

So the slope in the $\boldsymbol{u} = \frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{j}$ direction is

$$\begin{aligned} = \frac{1}{\sqrt{2}}(-2) + \frac{1}{\sqrt{2}}(-8) \quad &= \quad \frac{-10}{\sqrt{2}} \\ &= \quad -5\sqrt{2}. \end{aligned}$$

The rate of change of $f$ in the direction $\boldsymbol{u}$ is called a *directional derivative.*

**Directional Derivatives.**

The directional derivative in the direction $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$, where $\|\boldsymbol{u}\| = 1$, at $(a,\, b)$ is

$$f_{\boldsymbol{u}}(a,\, b) = \frac{\partial f}{\partial x}(a,\, b)u_1 + \frac{\partial f}{\partial y}(a,\, b)u_2$$

and it is simply the slope of the surface $f(x,\, y)$ in the direction $\boldsymbol{u}$. (But remember $\boldsymbol{u}$ must be a unit vector.)

**Example**. If $f(x, y) = x^2 - 3y^2 + 6y$. Find the slope at $(1, 0)$ in the direction $\boldsymbol{i} - 4\boldsymbol{j}$.

Now $\|\boldsymbol{i} - 4j\| = \sqrt{1 + 16} = \sqrt{17}$.

So $\boldsymbol{u} = \dfrac{1}{\sqrt{17}}(\boldsymbol{i} - 4\boldsymbol{j})$ is a unit vector in the direction $\boldsymbol{i} - 4\boldsymbol{j}$.

Now slope $= f_{\boldsymbol{u}}(1, 0) = \dfrac{\partial f}{\partial x}(1, 0)\dfrac{1}{\sqrt{17}} + \dfrac{\partial f}{\partial y}(1, 0)\left(\dfrac{-4}{\sqrt{17}}\right)$,

$$\frac{\partial f}{\partial x}(1, 0) = 2x\Big|_{(1,0)} = 2$$

and

$$\frac{\partial f}{\partial y}(1, 0) = (-6y + 6)\Big|_{(1,0)} = 6.$$

So slope $= f_{\boldsymbol{u}}(1, 0) = \dfrac{2}{\sqrt{17}} - \dfrac{6.4}{\sqrt{17}} = -\dfrac{22}{\sqrt{17}}$.

**The Gradient Vector**

Another way to picture the directional derivative in the direction $\boldsymbol{u}$

$$f_{\boldsymbol{u}}(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2$$

is to think of the gradient, or slope, as a vector itself.

$$\text{grad}\, f(a, b) = \nabla f(a, b) = f_x(a, b)\boldsymbol{i} + f_y(a, b)\boldsymbol{j}.$$

For example if $f(x, y) = x^2 - 3(y - 1)^2 + 3$

$$\nabla f = 2x\boldsymbol{i} - 6(y - 1)\boldsymbol{j}$$

or

$$\nabla f(1, 0) = 2\boldsymbol{i} + 6\boldsymbol{j}.$$

Then the directional derivative is just the dot product of $\nabla f$ with $\boldsymbol{u}$ since

$$\nabla f.\boldsymbol{u} = (f_x\boldsymbol{i} + f_y\boldsymbol{j}).(u_1\boldsymbol{i} + u_2\boldsymbol{j}) = f_x u_1 + f_y u_2$$

$$\Rightarrow f_{\boldsymbol{u}}(a, b) = \nabla f(a, b).\boldsymbol{u}$$

and this is a much easier formula to use.

**Example**. If $g(x,\, y) = e^{x^2} \cos y$ find the directional derivative of $g(x,\, y)$ at $(1,\, \pi)$ in the direction $(-3\boldsymbol{i} + 4\boldsymbol{j})$.

First we need to find the unit vector in the given direction. Now $\| -3\boldsymbol{i} + 4\boldsymbol{j} \| = \sqrt{9+16} = 5$. So $\boldsymbol{u} = \frac{-3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$.

Next find the gradient vector at $(1,\, \pi)$:

$$\begin{aligned} \nabla g &= \frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} \\ &= 2xe^{x^2} \cos y \boldsymbol{i} - e^{x^2} \sin y \boldsymbol{j} \\ \nabla g(1,\, \pi) &= 2e^1(-1)\boldsymbol{i} - 0\boldsymbol{j} = -2e\boldsymbol{i} \\ g_{\boldsymbol{u}}(1,\, \pi) &= \nabla g(1,\, \pi).\left(\frac{-3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}\right) \\ &= \frac{6e}{5}. \end{aligned}$$

**Example**. Find the slope of $f(x,\, y) = 1 - x^2 - y^2$ at $(0,\, 1)$ in the direction $(\boldsymbol{i} - \boldsymbol{j})$. First $\boldsymbol{u} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$ is the unit vector in the given direction.

$$\nabla f = -2x\boldsymbol{i} - 2y\boldsymbol{j} \Rightarrow \nabla f(1,\, 0) = -2\boldsymbol{i}.$$

So slope is $f_{\boldsymbol{u}} = \nabla f.\boldsymbol{u} = +\frac{2}{\sqrt{2}} = \sqrt{2}$. The gradient vector provides an easy way to calculate the slope in any direction but can we give it a geometrical interpretation?

Consider the contour diagram of a plane $z = f(x,\, y) = mx + ny + c$.
The contours $y = \frac{-m}{n}x + \frac{z_0 - c}{n}$ have slope $\frac{-m}{n}$ in the $(x,\, y)$ plane.

The gradient vector, $m\boldsymbol{i} + n\boldsymbol{j}$, is perpendicular to the contours. Also it points in the direction of increasing $f$. In fact the direction in which it points is the direction of greatest slope.

But what if $f(x,\, y)$ is not a plane.
Consider $f(x,\, y) = x^2 + y^2$. The contours are circles. $\nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j}$ which points radially out. Once again $\nabla f$ points in the direction of greatest slope, perpendicular to the contour lines.

**Properties of the Gradient Vector $\nabla f$**

The direction of $\nabla f(a, b)$ is perpendicular to the contour line through $(a, b)$ and in the direction of increasing $f$. In fact the direction and magnitude of steepest slope at $(a, b)$ is given by $\nabla f(a, b)$.

**Example**. $T(x, y) = 20 - 4x^2 - y^2$ describes the temperature on the surface of a metal plate. $x$ and $y$ are in cm and $T$ is in °C.
In what direction from $(2, -3)$ does the temperature increase most rapidly?
The direction is simply $\nabla T = \frac{\partial T}{\partial x}\boldsymbol{i} + \frac{\partial T}{\partial y}\boldsymbol{j} = -8x\boldsymbol{i} - 2y\boldsymbol{j}$. So

$$\nabla T(2, -3) = -16\boldsymbol{i} + 6\boldsymbol{j}$$

.

The direction **in terms of angles** is $\pi - \arctan\left(\frac{6}{16}\right)$.
**Example**. A team of oceanographers are mapping the ocean floor to assist in the recovery of a sunken ship. Using sonar they develop the model

$$D = 250 - 30x^2 - 50\sin\left(\frac{\pi y}{2}\right),$$

where $x$ and $y$ are distance in km, $D$ is depth in meters, and $-2 \le x \le 2$ and $-2 \le y \le 2$.

(a) Change the model to obtain a graph of the ocean floor.

Let $h(x, y)$ be the height above -250 below sea level. Then $D + h = 250$ and

$$h(x, y) = 30x^2 + 50\sin\left(\frac{\pi y}{2}\right).$$

(b) The ship is located at $(1, 0.5)$. What is its depth?

$$D(1, 0.5) = 250 - 30 - 50\sin\frac{\pi}{4} \simeq 184.6\text{m}.$$

(c) Determine the steepness of the ocean floor in the positive $x$ direction and in the positive $y$ direction. Finally, determine the magnitude and direction of greatest rate of change of depth from the position of the ship.

Slope in the $x$ direction is

$$\frac{\partial h}{\partial x}(1, 0.5) = 60x\Big|_{(1,0.5)} = 60.$$

But we must be careful here because $h$ is in meters while $x$ and $y$ are in kilometers. So in fact the slope is $\frac{60}{1000} = 0.06$.
Slope in the $y$ direction is

$$\begin{aligned} \frac{\partial h}{\partial y}(1, 0.5) &= \frac{50\pi}{2} \cos \frac{\pi y}{2}\Big|_{(1,0.5)} = 25\pi \cos \frac{\pi}{4} \\ &= \frac{25}{\sqrt{2}}\pi. \end{aligned}$$

So slope is $\frac{25}{\sqrt{2}.1000}\pi = \frac{\pi}{40\sqrt{2}}$.
The direction of greatest rate of change is given by

$$\nabla h = \frac{60\boldsymbol{i} + \frac{25\pi}{\sqrt{2}}\boldsymbol{j}}{1000}$$

or $\arctan\left(\frac{25\pi}{60\sqrt{2}}\right) = \arctan\left(\frac{5\pi}{12\sqrt{2}}\right)$ and the magnitude is $\frac{\sqrt{3600 + \frac{625\pi^2}{2}}}{1000}$.