MT152

2. Partial Derivative and Tangent Planes

2.5. The Chain Rule and Applications

The chain rule is used for differentiating a function of a function such as e^{x^2}. To evaluate e^{x^2}, we first square x and then take the exponential of the result $y = f(u(x))$ where $f(u) = e^u$ and $u(x) = x^2$.

$$\frac{dy}{dx} = \frac{df}{du}\frac{du}{dx} = e^u.2x = 2xe^{x^2}.$$

Example. Suppose the radius of a cylinder is decreasing at a rate $\dfrac{dr}{dt} = -2$ cm/s. How fast is the volume decreasing when $r = 1$ m and $h = 2$ m?

Volume of a cylinder is given by $V = \pi r^2 h$. Now sine h is constant it follows that

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 2\pi r h\frac{dr}{dt} = 2\pi 100.200.(-2) = -80,000\pi \text{ cm}^3/\text{sec}.$$

What if h is a function of t as well?

Chain Rule for functions of two variables

Suppose $f(x(t),\, y(t))$ and we want $\dfrac{df}{dt}$:

$$\frac{df}{dt} = \lim_{\Delta t\to 0}\frac{\Delta f}{\Delta t} = \lim_{\Delta t\to 0}\frac{f(x(t+\Delta t),\, y(t+\Delta t)) - f(x(t),\, y(t))}{\Delta t}.$$

Now if f is smooth and Δf is small we can relate it to Δx and Δy through the linear approximation;

$$\begin{aligned} \Delta f &\simeq f_x\Delta x + f_y\Delta y \\ \Rightarrow \quad \frac{\Delta f}{\Delta t} &\simeq f_x\frac{\Delta x}{\Delta t} + f_y\frac{\Delta y}{\Delta t}. \end{aligned}$$

Then we can let $\Delta t \to 0$ and if $x(t)$ and $y(t)$ are smooth too then

$$\frac{df}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$$

which is the Chain Rule for $f(x(t),\, y(t))$.

Example. What if $h(t)$ is also decreasing, $\frac{dh}{dt}(t) = -1\,\text{cm/s}$, and as before $\frac{dr}{dt} = -2\,\text{cm/s}$.

In this case we have

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{dr}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \\ &= 2\pi r h\frac{dr}{dt} + \pi r^2\frac{dh}{dt} \quad = 2\pi.100.200.(-2) + \pi(100)^2(-1) \\ &\qquad\qquad\qquad\qquad = -90,000\pi\,\text{cm}^3/\text{sec}.\end{aligned}$$

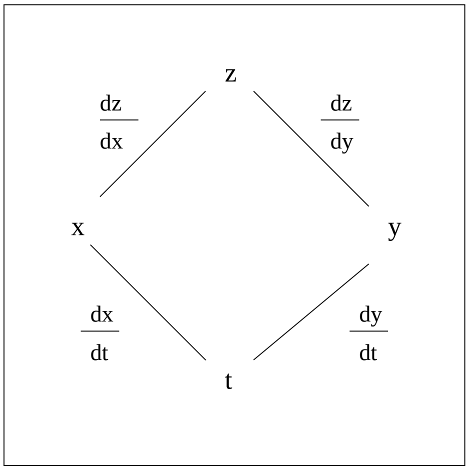
In fact the chain rule can be extended to any number of dimensions eg.

If $V(a(t)),\, b(t),\, c(t)) = abc$ is volume of a box then

$$\begin{aligned}\frac{dV}{dt} &= V_a\frac{da}{dt} + V_b\frac{db}{dt} + V_c\frac{dc}{dt} \\ &= bc\frac{da}{dt} + ac\frac{db}{dt} + ab\frac{dc}{dt}.\end{aligned}$$

Tree diagrams

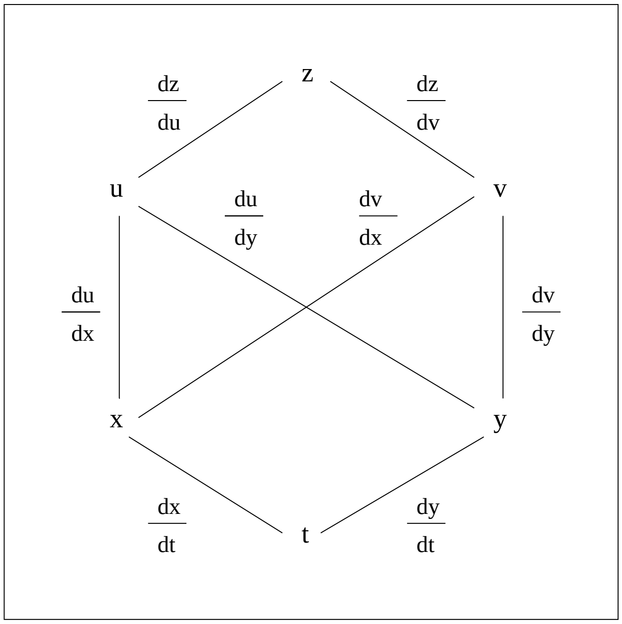
To keep check of what depends on what, you can draw a tree diagram for $z = f(x(t), y(t))$ as follows.



So

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Or if $z = f\left(u(x(t),\, y(t)),\; v(x(t),\, y(t))\right)$ then the tree diagram is



So

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right) \\ &= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\frac{dy}{dt}\end{aligned}$$

From which we see that since z can also be thought of as a function of $x(t)$ and $y(t)$ so that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}, \\ \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.\end{aligned}$$

Example. Find the slope of $z = f(x,\, y) = \sin x \cos y$ in the radial direction at $(\pi,\, 2\pi)$.

First recall polar coordinates

$$\begin{matrix} x = r\cos\theta & & r = x^2 + y^2 \\ & \Rightarrow & \\ y = r\sin\theta & & \tan\theta = y/x \end{matrix}$$

The slope in the radial direction is $\dfrac{\partial f}{\partial r}$ where $f(x(r,\, \theta),\; y(r,\, \theta))$.

So

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}.$$

Now $\frac{\partial f}{\partial x} = \cos x \cos y$, $\frac{\partial f}{\partial y} = -\sin x \sin y$, $\frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial y}{\partial r} = \sin\theta$.
So at $(x, y) = (\pi, 2\pi)$,

$$\begin{aligned} \frac{\partial f}{\partial x}(\pi, 2\pi) &= (-1)(1) = -1 \\ \frac{\partial f}{\partial y}(\pi, 2\pi) &= (-0)(0) = 0. \end{aligned}$$

Now since $\tan\theta = \frac{y}{x}\Big|_{(\pi,2\pi)} = 2$ we obtain

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{5}}.$$

Similarly,

$$\frac{\partial y}{\partial r} = \sin\theta = \sqrt{1-\cos^2\theta} = \frac{2}{\sqrt{5}}.$$

Finally, $\frac{\partial f}{\partial r} = (-1)(\frac{1}{\sqrt{5}}) + 0(\frac{2}{\sqrt{5}} = \frac{-1}{\sqrt{5}}$.

Example. Show that any function of the form $z = f(x+at) + g(x-at)$ satisfies $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$.

Let $u(x,t) = (x+at)$ and $v(x,t) = (x-at)$. Then $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial t} = a$, $\frac{\partial v}{\partial x} = 1$ and $\frac{\partial v}{\partial t} = -a$.
Now $z(u(x, t), v(x, t)) = f(u) + g(v)$. So

$$\begin{aligned} \frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} &= \frac{df}{du}\frac{\partial u}{\partial t} + \frac{dg}{dv}\frac{\partial v}{\partial t} \\ &= a(f'(u) - g'(v)). \end{aligned}$$

Since $\frac{\partial z}{\partial t}$ is a function of $u(x, t)$ and $v(x, t)$ we can differentiate it with respect to t;

$$\frac{\partial^2 z}{\partial t^2} = a(\frac{d^2 f}{du^2}\frac{\partial u}{\partial t} - \frac{d^2 g}{dv^2}\frac{\partial v}{\partial t}) = a^2(f'' + g'').$$

Similarly, $\frac{\partial^2 z}{\partial x^2} = (f'' + g'')$. So

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}.$$

The above equation is called the *wave equation.*

For example, the travelling wave $z = A + B\cos(x + \frac{1}{2}t)$ is a solution to the wave equation

$$\frac{\partial^2 z}{\partial t^2} = \frac{1}{4}\frac{\partial^2 z}{\partial x^2}.$$