3. Max and Min Problems on Surfaces

3.0 Introduction and some examples

For functions of one variable $y = f(x)$, which are *smooth*, local maxima and minima occur where $f(x)$ has zero slope ie. $\dfrac{df}{dx} = 0$.

For example

$$\begin{aligned} y &= f(x) = x^2 - x^4 \\ \frac{dy}{dx} &= 2x - 4x^3 = 2x(1-2x^2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = \pm\frac{1}{\sqrt{2}}. \end{aligned}$$

Then the second derivative test tells us if $f(x)$ is concave up $\Rightarrow$ min or if it is concave down $\Rightarrow$ max.

Here $\dfrac{d^2y}{dx^2} = 2 - 12x^2$. Since $\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = 2 > 0 \Rightarrow$ there is a local min at $x = 0$.

Since $\left.\dfrac{d^2y}{dx^2}\right|_{x=\pm\frac{1}{\sqrt{2}}} = -4 < 0 \Rightarrow x = \dfrac{1}{\sqrt{2}}$ and $x = \dfrac{-1}{\sqrt{2}}$ and are both local maxima. Note however that the local min is not a global min. At the local minimum $x = 0$ and $f(0) = 0$. But for $|x| > 1$ we have $f(x) < 0$.

If we define the domain as $[-2, 2]$ the *global* min occurs on the boundary at $f(\pm 2) = 4 - 16 = -8$. The global max, however is at the local max, ie. $f(\frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

The global max or min of a continuous function $f(x)$ occurs either at a local min or max or on the boundary.

While it is not really relevant to consider discontinuous functions cusps, where $\dfrac{df}{dx}$ is undefined, there may be local maxima or minima.

Consider $f(x) = |x - 3|$. It has a local min at $x = 3$ but $f(x)$ is not smooth at $x = 3$. The derivative of $f(x)$ is not defined at $x = 3$ because

$$\lim_{x\to 3_-} \frac{\Delta f(x)}{\Delta x} = -1 \neq \lim_{x\to 3^+} \frac{\Delta f}{\Delta x} = 1.$$

So we need a more general definition of a *local* min or max. A local min occurs at a point x_0 where $f(x) > f(x_0)$ for all x near to x_0.

A local max occurs at a point x_0 where $f(x) < f(x_0)$ for all x near to x_0.

If $f(x)$ is continuous a local max or min occurs either where $\frac{df}{dx} = 0$ or where $\frac{df}{dx}$ is undefined.

A function of two variables $f(x, y)$ can also be undifferentiable, if either $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ is undefined. For example $z = f(x, y) = 5 - \sqrt{x^2 + y^2}$ is undifferentiable at $(0, 0)$, and it has a local max there.

So we will have a similar definition of a local max or local min. But if $f(x, y)$ is differentiable a local max or min will occur where the slope in the x direction $\frac{\partial f}{\partial x} = 0$ **and** the slope in the y direction $\frac{\partial f}{\partial y} = 0$. Now whether it is a max or min depends on the second order derivatives.

Example. $f(x, y) = \left(\frac{1}{2} - x^2 + y^2\right) e^{(1-y^2-x^2)}$.