

3. Max and Min Problems on Surfaces

3.1 Second Order Partial Derivatives and Quadratic Approximations

In functions of one variable if $\frac{df}{dx}(a) = 0$ the second order derivative, the curvature, determines if $x = a$ is a max or min.

But for functions of two variables it is not so simple. Take

$z = f_1(x, y) = x^2 + y^2$, it has a min at $(0, 0)$ and $z = f_2(x, y) = -x^2 - y^2$ has a max at $(0, 0)$, **but** $z = f_3(x, y) = x^2 - y^2$ has a saddle at $(0, 0)$,

which is neither a local max nor min even though $\frac{\partial f_3}{\partial x}(0, 0) = 0$ and $\frac{\partial f_3}{\partial y}(0, 0) = 0$! To understand this we need to be able to approximate the surface close to $(0, 0)$ with a Quadratic Function. The linear approximation to $f(x, y)$ at $(0, 0)$ is

$$L(x, y) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y.$$

The Quadratic approximation to $f(x, y)$ at $(0, 0)$ is a quadratic function of x and y :

$$Q(x, y) = f(0, 0) + mx + ny + ax^2 + bxy + cy^2.$$

But what are the coefficients m, n, a, b and c ? If $f(x, y) = Q(x, y) = f(0, 0) + mx + ny + ax^2 + bxy + cy^2$ then $\frac{\partial f}{\partial x} = m + 2ax + by \Rightarrow \frac{\partial f}{\partial x}(0, 0) = m$ and $\frac{\partial f}{\partial y} = n + bx + 2cy \Rightarrow \frac{\partial f}{\partial y}(0, 0) = n$. Moreover,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 2a &\Rightarrow & a = \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial^2 f}{\partial x \partial y} &= b &\Rightarrow & b = \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y^2} &= 2c &\Rightarrow & c = \frac{1}{2} \frac{\partial^2 f}{\partial y^2}. \end{aligned}$$

In fact in general

$$\begin{aligned} a &= \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0, 0) \\ b &= \frac{\partial^2 f}{\partial x \partial y}(0, 0) \\ c &= \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(0, 0). \end{aligned}$$

(You can use Taylor Series to prove this.)

So that the Quadratic approximation to $f(x, y)$ near (a, b) is

$$Q(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \\ + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(a, b)(x - a)^2 + \frac{\partial^2 f}{\partial x \partial y}(a, b)(x - a)(y - b) + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(a, b)(y - b)^2.$$

For example let's work out the quadratic approximation of

$$f(x, y) = 1 - x^2 + y^2 + xy + x^3 + x^2y^2$$

about $(0, 0)$ because

$$\begin{aligned} \frac{\partial f}{\partial x}(0, 0) &= -2x + y + 3x^2 + 2xy^2 \Big|_{x=0, y=0} = 0 \\ \frac{\partial f}{\partial y}(0, 0) &= 2y + x + 2yx^2 \Big|_{(0,0)} = 0 \\ \frac{\partial^2 f}{\partial x^2}(0, 0) &= -2 + 6x + 2y^2 \Big|_{(0,0)} = -2 \\ \frac{\partial^2 f}{\partial y \partial x}(0, 0) &= 1 + (2x)(2y) \Big|_{(0,0)} = 1 \\ \frac{\partial^2 f}{\partial y^2}(0, 0) &= 2 + 2x^2 \Big|_{(0,0)} = 2 \end{aligned}$$

and $f(0, 0) = 1$.

$$Q(x, y) = 1 + 0x + 0y + \frac{-2}{2}x^2 + 1xy + \frac{2}{2}y^2 = 1 - x^2 + xy + y^2 \\ f(x, y) \simeq Q(x, y) \quad \text{for} \quad (x, y) \simeq (0, 0).$$

If $|x|, |y| < 0.1$ then we are ignoring terms on the order of $(0.1)^3 = 0.001$.

In this example the function is a polynomial in x and y and we can almost read off the quadratic approximation. But in the next example we really need to use the formula.

Example. Find the linear and quadratic approximation of $z = f(x, y) = e^{-x^2-y^2}$ about

(0, 0)

$$\begin{aligned}\frac{\partial f}{\partial x} &= -2xe^{-x^2-y^2} \Rightarrow \frac{\partial f}{\partial x}(0, 0) = 0, & \frac{\partial f}{\partial y} &= -2ye^{-x^2-y^2} \Rightarrow \frac{\partial f}{\partial y}(0, 0) = 0 \\ \frac{\partial^2 f}{\partial x^2} &= (-2 + 4x^2)e^{-x^2-y^2} \Rightarrow \frac{\partial^2 f}{\partial x^2}(0, 0) = -2 \\ \frac{\partial^2 f}{\partial x \partial y} &= 4xye^{-x^2-y^2} \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(0, 0) = 0 \\ \frac{\partial^2 f}{\partial y^2} &= (-2 + 4y^2)e^{-x^2-y^2} \Rightarrow \frac{\partial^2 f}{\partial y^2}(0, 0) = -2.\end{aligned}$$

So the linear approx at (0, 0) is $L(x, y) = 1$. (ie. the tangent plane is $z = 1$) and the quadratic approximation at (0, 0) is

$$Q(x, y) = 1 + -2x^2 - 2y^2$$

from which we can see that the origin is a max.

You can calculate the Quadratic approximation at any point (a, b) . Just use the formula.

Example. Find the quadratic approximation to $z = f(x, y) = x^2y$ at $(1, 2)$.

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2xy \Rightarrow \frac{\partial f}{\partial x}(1, 2) = 4 \\ \frac{\partial f}{\partial y} &= x^2 \Rightarrow \frac{\partial f}{\partial y}(1, 2) = 1 \\ \frac{\partial^2 f}{\partial x^2} &= 2y \Rightarrow \frac{\partial^2 f}{\partial x^2}(1, 2) = 4 \\ \frac{\partial^2 f}{\partial x \partial y} &= 2x \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(1, 2) = 2 \\ \frac{\partial^2 f}{\partial y^2} &= 0.\end{aligned}$$

So $Q(x, y) = 2 + 4(x - 1) + (y - 2) + \frac{4}{2}(x - 1)^2 + 2(x - 1)(y - 2)$ and $f(x, y) \simeq Q(x, y)$ for $(x, y) \simeq (1, 2)$.

The quadratic approximation can be used in applications to get a better estimate of error.

Example. Let's take that barrel again, base radius 1, height 2

$$\begin{aligned}V(r, h) &= \pi r^2 h \Rightarrow V(1, 2) = 2\pi \\ \frac{\partial V}{\partial r} &= 2\pi r h \Rightarrow \frac{\partial V}{\partial r}(1, 2) = 4\pi \\ \\ \frac{\partial V}{\partial h} &= \pi r^2 \Rightarrow \frac{\partial V}{\partial h}(1, 2) = \pi \\ \\ \frac{\partial^2 V}{\partial r^2} &= 2\pi h \Rightarrow \frac{\partial^2 V}{\partial r^2}(1, 2) = 4\pi \\ \\ \frac{\partial^2 V}{\partial r \partial h} &= 2\pi r \Rightarrow \frac{\partial^2 V}{\partial r \partial h}(1, 2) = 2\pi \\ \\ \frac{\partial^2 V}{\partial h^2} &= 0.\end{aligned}$$

So

$$\begin{aligned}V(r, h) \simeq Q(r, h) &= 2\pi + 4\pi(r - 1) + \pi(h - 2) \\ &\quad + \frac{4\pi}{2}(r - 1)^2 + 2\pi(r - 1)(h - 2) + 0\end{aligned}$$

for $(r, h) \simeq (1, 2)$.

Let $\Delta r = r - 1$ and $\Delta h = h - 2$. Now if $\Delta r = 0.05$ and $\Delta h = 0.1$ what is the approximate error in volume ΔV ?

Last time we used linear approximations to get 0.3π . Using quadratic approximations

$$\begin{aligned}\Delta V \simeq Q(r, h) - 2\pi &= 4\pi\Delta r + \pi\Delta h + 2\pi\Delta r^2 + 2\pi\Delta r\Delta h \\ &= \pi(4(0.05) + 0.1 + 2(0.05)^2 + 2(0.05)(0.1)) \\ &= \pi(0.2 + 0.1 + 2(0.0025 + 0.005)) \\ &= 0.315\pi \quad (\text{actual } 0.31515).\end{aligned}$$