3. Max and Min Problems on Surfaces

3.2 Critical Points, local max and min and saddle points

First we define a *local max* and *min*.

- $f(x, y)$ has a *local max* at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) near to (a, b); and
- $f(x, y)$ has a *local min* at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) near to (a, b).

Now if f is differentiable a local max or min occurs where the tangent plane is horizontal, ie. where $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$.
We can put this more succinctly by using the gradient;

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}.$$

Remark. $\nabla f = \mathbf{0} \Leftrightarrow \dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0 \Leftrightarrow$ the tangent plane at (a, b) is horizontal.
Also if ∇f is undefined $\Leftrightarrow$ either $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial f}{\partial y}$ is undefined. So if we define:

Definition. A *critical point* of $z = f(x, y)$ is a point where either $\nabla f = 0$ or ∇f is undefined.
Then the local max and min will occur at critical points, but not all critical points are local max or min because they may be saddles.

Example. Find the critical points of $z = f(x, y) = x^2 + 3x + y^2$.

$$\nabla f = (2x+3)\boldsymbol{i} + 2y\boldsymbol{j} = \mathbf{0} \Rightarrow 2x + 3 = 0 \quad \text{and} \quad y = 0 \Rightarrow x = \frac{-3}{2},\ y = 0.$$

By completing the square $z = (x + \dfrac{3}{2})^2 + y^2 - \frac{9}{4}$ we can see that $(\dfrac{-3}{2}, 0)$, where $f(\frac{-3}{2}, 0) = \frac{-9}{4}$ is a local and global min.

Example. Find the critical points of

$$z = f(x, y) = x^2 + 3x - y^2$$

$$\nabla f = (2x+3)i - 2yj \Rightarrow (\frac{-3}{2}, 0)$$

is a critical point.

But now $z = (x+\frac{3}{2})^2 - y^2 - \frac{9}{4}$ is a saddle. So $(\frac{-3}{2}, 0)$ is not a local min or a local max.

Definition. A function f has a saddle point P_0 if P_0 is a critical point of f and if within any distance of P_0 (no matter how small) there are points P_1 and P_2 with

$$f(P_1) > f(P_0) \quad \text{and} \quad f(P_2) < f(P_0).$$

Example. Find all the critical points of

$$f(x, y) = \left(\frac{1}{2} - x^2 + y^2\right) e^{(1-y^2-x^2)}.$$

Then use a quadratic approximation to investigate the curvature.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -2xe^{1-y^2-x^2} + \left(\frac{1}{2} - x^2 + y^2\right)(-2x)e^{(1-y^2-x^2)} \\
&= -2x\left(\frac{3}{2} - x^2 + y^2\right)e^{(1-y^2-x^2)} \\
\frac{\partial f}{\partial y} &= 2ye^{(1-y^2-x^2)} + (\frac{1}{2} - x^2 + y^2)(-2y)e^{(1-y^2-x^2)} \\
&= -2y(-\frac{1}{2} - x^2 + y^2)e^{(1-y^2-x^2)}
\end{aligned}$$

$$\nabla f = 0 \Rightarrow \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0.$$

$$\Rightarrow x(\frac{3}{2} - x^2 + y^2) = 0 \quad \text{and} \quad y(-\frac{1}{2} - x^2 + y^2) = 0$$

$$\Rightarrow \{x = 0 \text{ and } y = 0\} \text{ or } \left\{x = 0 \text{ and } -\frac{1}{2} - x^2 + y^2 = 0\right\}$$

$$\text{or } \left\{\frac{3}{2} - x^2 + y^2 = 0 \text{ and } y = 0\right\} \text{ or } \left\{\frac{3}{2} - x^2 + y^2 = 0 \text{ and } -\frac{1}{2} - x^2 + y^2 = 0\right\}$$

So the critical points are

$$(1) \quad (0, 0), \qquad (2) \quad (0, \pm\sqrt{\frac{1}{2}}), \qquad (3) \quad (\pm\sqrt{\frac{3}{2}}, 0).$$

We note that $\frac{3}{2} - x^2 + y^2 = 0$ and $-\frac{1}{2} - x^2 + y^2 = 0$ is impossible.

Now

$$\frac{\partial^2 f}{\partial x^2} = \left(\left(-2\left(\frac{3}{2} - x^2 + y^2\right) - 2x(-2x)\right) - 2x(-2x)\left(\frac{3}{2} - x^2 + y^2\right)\right) e^{1-y^2-x^2}$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2}(0,\, 0) = -3e,\ \frac{\partial^2 f}{\partial x^2}\left(0,\, \pm\sqrt{\frac{1}{2}}\right) = -4e^{\frac{1}{2}},\ \frac{\partial^2 f}{\partial x^2}\left(\pm\sqrt{\frac{3}{2}},\, 0\right) = 6e^{-\frac{1}{2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \left(-2x(2y) - 2x\left(\frac{3}{2} - x^2 + y^2\right)(-2y)\right) e^{1-y^2-x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,\, 0) = 0\,,\ \frac{\partial^2 f}{\partial x \partial y}\left(0,\, \pm\sqrt{\frac{1}{2}}\right) = 0\,,\ \frac{\partial^2 f}{\partial x \partial y}\left(\pm\sqrt{\frac{3}{2}},\, 0\right) = 0$$

$$\frac{\partial^2 f}{\partial y^2} = \left(-2\left(-\frac{1}{2} - x^2 + y^2\right) - 2y(2y) - 2y\left(-\frac{1}{2} - x^2 + y^2\right)(-2y)\right) e^{1-y^2-x^2}$$

$$\frac{\partial^2 f}{\partial y^2}(0,\, 0) = e\,,\ \frac{\partial^2 f}{\partial y^2}\left(0,\, \pm\sqrt{\frac{1}{2}}\right) = -2e^{\frac{1}{2}},\ \frac{\partial^2 f}{\partial y^2}\left(\pm\sqrt{\frac{3}{2}},\, 0\right) = 4e^{-\frac{1}{2}}.$$

Quadratic approximation at $(0,\, 0)$:

$$Q(x,\, y) = \frac{e}{2}(1 - 3x^2 + y^2).$$

The point $(0,\, 0)$ is a **saddle** point.

Quadratic approximation at $(0,\, \pm\sqrt{\frac{1}{2}})$:

$$Q(x,\, y) = e^{\frac{1}{2}}(1 - 2x^2 - (y - (\pm\sqrt{\frac{1}{2}}))^2).$$

The points $(0,\, \pm\sqrt{\frac{1}{2}})$ are **local max**.

Quadratic approximation at $(\pm\sqrt{\frac{3}{2}},\, 0)$:

$$Q(x,\, y) = e^{-\frac{1}{2}}(-1 + 3(x - (\pm\sqrt{\frac{3}{2}}))^2 + 2y^2).$$

The points $(\pm\sqrt{\frac{3}{2}},\, 0)$ are **local min**.

But what if $Q(x,\, y)$ has cross terms (xy).

For example $f(x,\, y) = x^2 + xy + y^2$ which has a critical point at $(0,\, 0)$ we can "sort of complete a square"

$$\begin{aligned} f(x,\, y) &= (x + \frac{1}{2}y)^2 - \frac{y^2}{4} + y^2 \\ &= (x + \frac{1}{2}y)^2 + \frac{3y^2}{4} \end{aligned}$$

So $f(x,y)$ has a local minimum at (0, 0).

Now consider the function

$$f(x,\,y) = x^2 + xy + \frac{y^2}{8} \;=\; (x+\frac{1}{2}y)^2 - \frac{y^2}{4} + \frac{y^2}{8}$$
$$= (x+\frac{1}{2}y)^2 - \frac{y^2}{8}.$$

This function has a saddle point at (0, 0).

But can we get a rule? Yes.

The Second Derivative test for max and min.

Say

$$\begin{aligned} f(x,\,y) &= ax^2 + bxy + cy^2 \\ &= a\left(\left(x+\frac{by}{2a}\right)^2 - \frac{b^2y^2}{4a^2} + \frac{c}{a}y^2\right) \\ &= a\left(\left(x+\frac{by}{2a}\right)^2 + \frac{(4ac-b^2)}{4a^2}y^2\right). \end{aligned}$$

Then if $4ac - b^2 > 0 \Rightarrow$ local max or min

and if $4ac - b^2 < 0 \Rightarrow$ saddle.

Further if $4ac - b^2 > 0$ then $\begin{cases} \text{if } a > 0 & \text{the function has a local min} \\ \text{if } a < 0 & \text{the function has a local max} \end{cases}$

Now if we use a quadratic approximation to the function at (0, 0)

$$\begin{aligned} Q(x,\,y) = f(0,\,0) + f_x(0,\,0)x \;&+\; f_y(0,\,0)y + f_{xx}(0,\,0)\frac{x^2}{2} \\ &+\; f_{xy}(0,0)xy + f_{yy}(0,\,0)\frac{y^2}{2} \end{aligned}$$

and if $\nabla f(0,0) = \mathbf{0} \Leftrightarrow f_x(0,\,0) = 0$ and $f_y(0,0,) = 0$.

Then if $f_{xx}(0,\,0)f_{yy}(0,\,0) - (f_{xy}(0,\,0))^2 < 0 \Rightarrow$ saddle

and if $f_{xx}(0,\,0)f_{yy}(0,\,0) - (f_{xy}(0,\,0))^2 > 0$ and $f_{xx}(0,0) > 0 \Rightarrow$ local min

and if $f_{xx}(0,\,0)f_{yy}(0,\,0) - (f_{xy}(0,\,0))^2 > 0$ and $f_{xx}(0,0) < 0 \Rightarrow$ local max.

The Second Derivative test for local max and min of $z = f(x,\,y)$ where $f(x,\,y)$ is smooth.

Let $(a,\,b)$ be a critical point of $f(x,\,y)$ ie. $\nabla f(a,\,b) = 0$.

Define $D = f_{xx}(a,\, b) f_{yy}(a,\, b) - (f_{xy}(a,\, b))^2$.

$$
\begin{array}{llll}
\text{If} & D < 0 & \Rightarrow \text{ saddle at } (a,\, b) & \\
\text{If} & D > 0 & \text{and } f_{xx}(a,\, b) > 0 & \Rightarrow \text{ local min at } (a,\, b) \\
\text{If} & D > 0 & \text{and } f_{xx}(a,\, b) < 0 & \Rightarrow \text{ local max at } (a,\, b) \\
\text{If} & D = 0 & \text{Help! could be anything.} &
\end{array}
$$

The beauty of this test is that we do not even have to write down the quadratic approximation.

Example. Find the critical points of $z = f(x,\, y) = e^{-x^2+y^2}$ and classify them as saddle, local max or min.

$$\frac{\partial f}{\partial x} = -2xe^{-x^2+y^2}, \quad \frac{\partial f}{\partial y} = 2ye^{-x^2+y^2}$$

$$\nabla f = 0 \Rightarrow x = y = 0$$

$$\frac{\partial^2 f}{\partial x^2} = (-2+4x^2)e^{-x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4xye^{-x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = (2+4y^2)e^{-x^2+y^2}.$$

So $D(0,0) = -2.2 - 0 < 0 \Rightarrow$ saddle at $(0,\, 0)$.

Only one critical point at $(0,\, 0)$ which is a saddle.

Example. Find and classify the critical points of $z = f(x,\, y) = x^2 + 4xy + \dfrac{y^3}{3}$.

$$\nabla f = (2x+4y)\boldsymbol{i} + (4x+y^2)\boldsymbol{j}$$

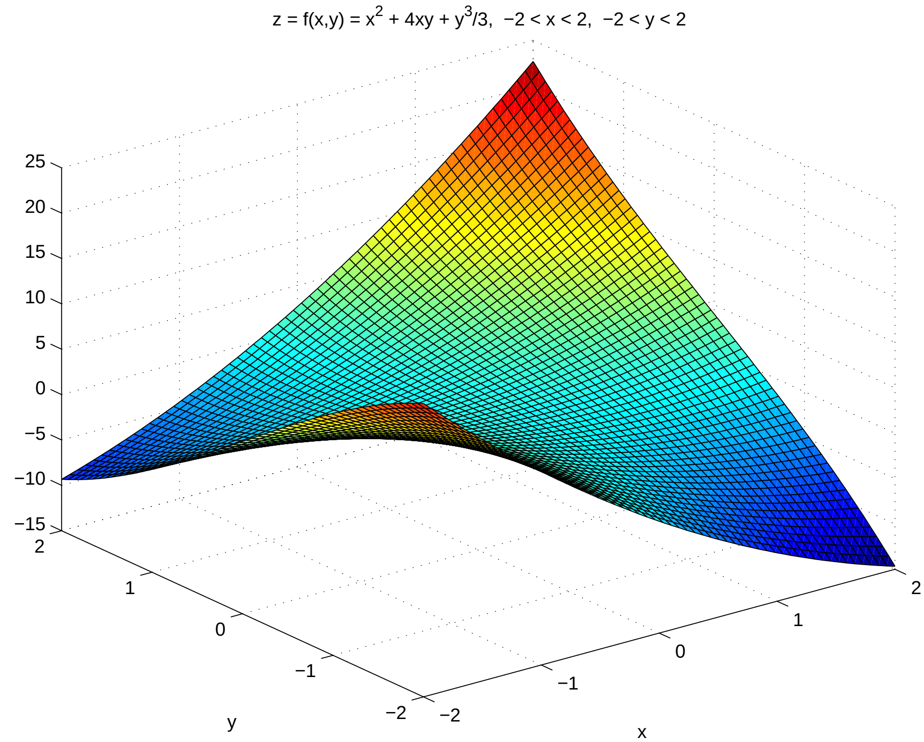
$$\nabla f = 0 \Rightarrow x = -2y \text{ and } 4x + y^2 = 0$$

So $-8y + y^2 = 0$ or $y(-8+y) = 0 \Rightarrow y = 0$ or 8. Therefore, the critical points are $(0,\, 0)$ and $(-16,\, 8)$. Now

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 4, \quad \frac{\partial^2 f}{\partial y^2} = 2y.$$

So $D(0,\, 0) = 2.0 - 4 < 0 \Rightarrow f(x,y)$ has a saddle point at $(0,0)$.

Moreover, $D(-16,\, 8) = 2.2.8 - 4 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0 \Rightarrow f(x,y)$ has a local min at $(-16,8)$.



Then we can use this to answer a question like:-

What is the maximum and minimum value of $z = f(x, y)$ on say $[-2, 2] \times [-2, 2]$?

The global max or min may occur at a critical point or on the boundary.

The global max is $f(2, 2) = \dfrac{68}{3}$ and the global min is $f(2, -2) = \dfrac{-44}{3}$ in that region.

Note that $f(-16, 8) = \dfrac{-256}{3}$ but the point $(-16, 8)$ is not in $[-2, 2] \times [-2, 2]$.

To work this out you have to take each boundary separately.

Say on $x = 2$ we have $f(2, y) = 4 + 8y + \dfrac{y^3}{3}$. We find the maximum or minimum value of the function on the boundary. Here $\dfrac{\partial f}{\partial y}(2, y) = 8 + y^2$ which is increasing so the max occurs at $f(2, 2) = \dfrac{68}{3}$ and min at $f(2, -2) = \dfrac{-44}{3}$. We also need to examine the boundaries $x = -2$, $y = 2$ and finally $y = -2$.