3. Max and Min Problems on Surfaces

3.3 Constrained Optimisation and Lagrange Multipliers

Often in a problem you need to maximise some function given that it satisfies a constraint.

For example. If x and y are numbers such that $x + y = 1$ what is the minimum value of $x^2 + y^2$.

Here we want to minimise $f(x,\, y) = x^2 + y^2$ subject to $g(x,\, y) = x + y = 1$. Sometimes you can solve for the constraint and substitute back;

Here $g(x,\, y) = 1 \Rightarrow y = 1 - x$ and

$$f(x, 1 - x) = F(x) = x^2 + (1 - x)^2 = 2x^2 - 2x + 1.$$

Now to find the min of $f(x)$ We proceed as follows.

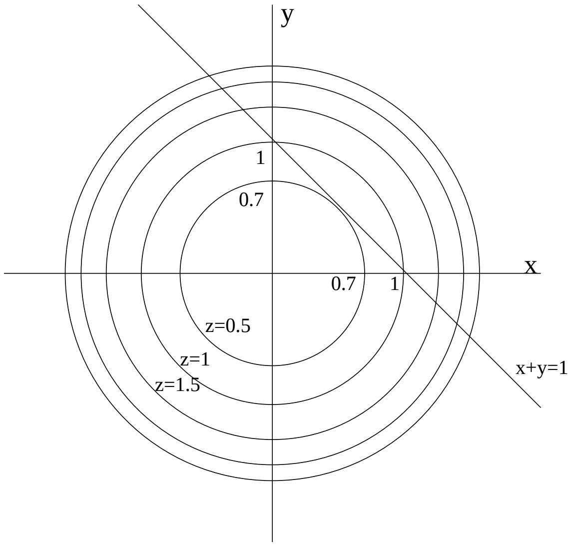
$$\frac{dF}{dx} = -2 + 4x = 0 \quad \Rightarrow \quad x = \frac{1}{2}.$$

Since $\dfrac{d^2F}{dx^2} = 4 > 0$ it follows that $F(x)$ has a minimum at $x = \dfrac{1}{2}$.

So $x^2 + y^2$ is minimum at $x = \dfrac{1}{2}$ and $y = \dfrac{1}{2}$. The minimum value is $= \dfrac{1}{2}$.

But there is a better way to do it which involves gradients. Think graphically!

Plot the contours of $f(x,\, y) = x^2 + y^2$ and the constraint $g(x,\, y) = 1$ on the same graph.



The minimum occurs where the two curves just touch which means that ∇f and ∇g are parallel. So that

$\nabla f = \lambda \nabla g$ for some constant λ - called the Lagrange Multipliers.

So to minimise $f(x, y)$ subject to $g(x, y) = 1$ we solve $\nabla f = \lambda \nabla g$ with $g(x, y) = 1$. Say $f(x, y) = x^2 + y^2$ and $g(x, y) = x + y = 1$. Then $\nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j}$ and $\nabla g = \boldsymbol{i} + \boldsymbol{j}$.

So $\nabla f = \lambda \nabla g \Rightarrow 2x\boldsymbol{i} + 2y\boldsymbol{j} = \lambda(\boldsymbol{i} + \boldsymbol{j}) \Rightarrow 2x = \lambda$ and $2y = \lambda \Rightarrow 2x = 2y \Rightarrow x = y$.

Now $g(x, y) = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$ and minimum value of $f(x, y)$ is

$$f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

This method can also be extended to any number of dimensions.

For example to minimise $x^2 + y^2 + z^2$ subject to $x + y + z = 1$, let $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = x + y + z = 1$.

Then

$$\begin{aligned}
\nabla f = \lambda \nabla g \quad &\Rightarrow \quad 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} = \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \\
&\Rightarrow \quad 2x = \lambda,\ 2y = \lambda,\ 2z = \lambda \\
&\Rightarrow \quad x = y = z. \quad \text{Now } g(x, x, x) = 3x = 1 \\
&\Rightarrow \quad x = y = z = \frac{1}{3}
\end{aligned}$$

and minimum value for f is $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$.

The Method of Lagrange multipliers

To optimise $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, solve

$$\nabla f = \lambda \nabla g \quad \text{with} \quad g(x, y, z) = c.$$

(λ is called the Lagrange Multiplier.)

Example. Find the maximum volume of a box resting on the xy- plane with one vertex at the origin and lying inside the plane $z + 2x + 3y = 6$.

We want to maximise $v = xyz$ subject to $g(x,\ y,\ z) = 2x + 3y + z = 6$.

$$\nabla f = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k} = \lambda(2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$$

$$\Rightarrow yz = 2\lambda,\ xz = 3\lambda \quad \text{and} \quad xy = \lambda$$

$$\Rightarrow 2xy = yz \quad \text{and} \quad 3xy = xz.$$

So either $y = 0$ (which would give zero volume ie. a min) or $2x = z$ and $x = 0$ (which would give zero volume ie. a min) or $3y = z$.
So for max $y = \dfrac{2x}{3}$ and $z = 2x$

$$\Rightarrow g(x,\ \frac{2x}{3},\ 2x) = 6 \Rightarrow 2x + 2x + 2x = 6 \Rightarrow x = 1 \Rightarrow y = \frac{2}{3} \Rightarrow z = 2.$$

Therefore, maximum volume is $1.\dfrac{2}{3}.2 = \dfrac{4}{3}$.