# 4. Differential Equations

## 4.0 Introduction

Differential equations are equations for some unknown, say $y(t)$, which involve the derivatives of $y(t)$.

**Example.** If $y$ is the distance covered by a car travelling at a constant speed of 60 km/h then $\frac{dy}{dt} = 60$. Here we can simply integrate to solve for $y(t)$;

$$\frac{dy}{dt} = 60 \quad \Rightarrow \quad \int dy = \int 60dt$$

$$\Rightarrow \quad y - y_0 = 60(t - t_0)$$

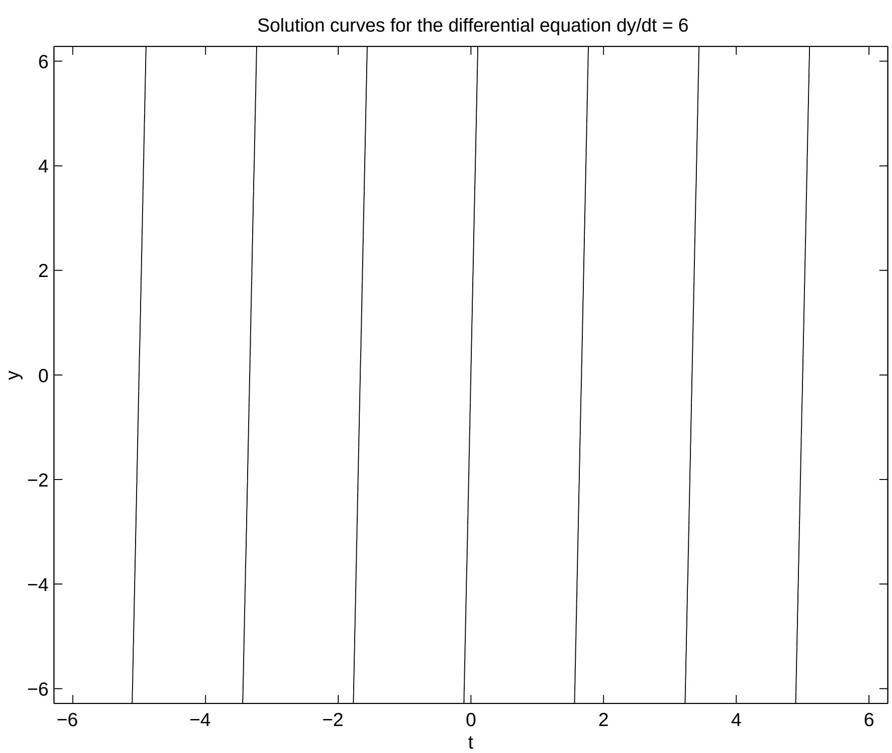
$$\text{or} \qquad y = 60t + c \quad (c = y_0 - 60t_0).$$

To solve for $c$ we need more information.

If say $y(0) = 10$ then $c = 10 - 60 \times 0 = 10 \quad \Rightarrow y = 60t + 10.$

### Solution Curves

For different values of $c$ you will get different solutions. Here we graph these solutions. If $c = 0$ then $y = 60t$ and all the other lines are parallel to this line.



The "curves" $y = 60t + c$ are called *solution curves* to the differential equation

$$\frac{dy}{dt} = 60.$$

They all have slope 60, just as the equation tells us.

To solve for a particular solution we need to specify some point that the solution curve must pass through. Often the point is an initial condition $y(0)$ then the problem;
Solve $\frac{dy}{dt} = 60$ with $y(0) = 2$ is called an *initial value problem.*
The solution is called a *particular solution*; $y = 60t + 2$ in this case.

**Example.** Solve for the particular solution to the initial value problem

$$\frac{dy}{dt} = 2\sqrt{t} \quad \text{with} \quad y(0) = 1.$$

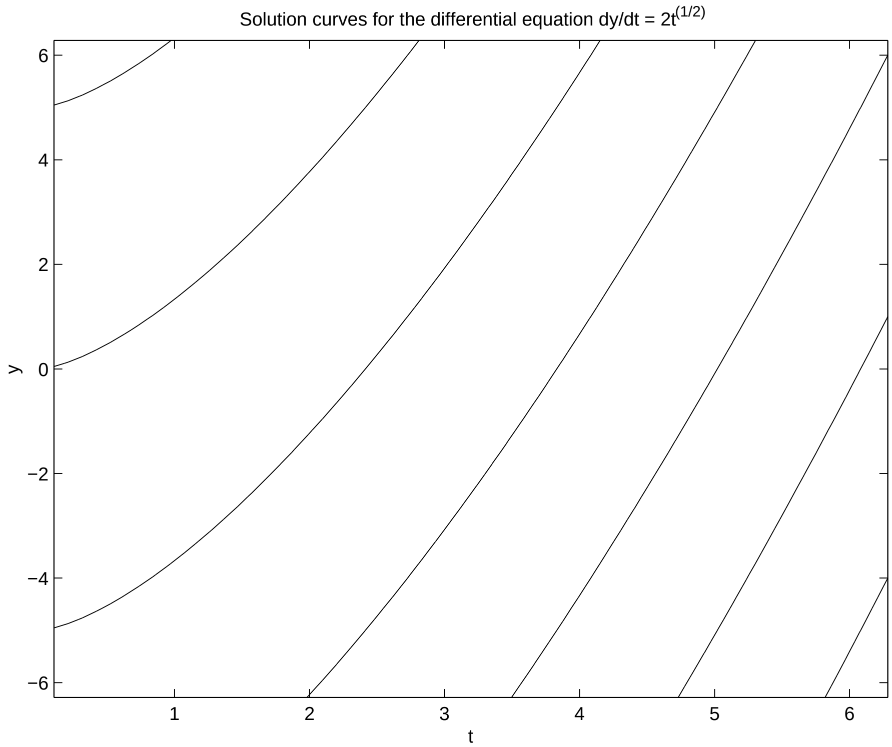
$$\frac{dy}{dt} = 2\sqrt{t} \quad \Rightarrow \quad \int dy = \int 2\sqrt{t}\, dt.$$

So $y = \frac{4t^{\frac{3}{2}}}{3} + c \Rightarrow y(0) = 1 = 0 + c$. Therefore, $y = \frac{4t^{\frac{3}{2}}}{3} + 1$ is the particular solution.

**Classification**

Equations involving only first order derivatives are called *first order differential equations*
eg $\quad \frac{dy}{dt} = 60.$

**Example**. Find and sketch the solution curves of $\frac{dy}{dt} = 2\sqrt{t}$. (This equation describes the rate at which ice forms in a pond).
As above, the general solution is $y = \frac{4t^{\frac{3}{2}}}{3} + c$. Note when $t = 1$ slope ($\frac{dy}{dt}$ at $t = 1$) is 2 and when $t = 4$ slope is $2\sqrt{4} = 4$, etc.




Equations involving second order derivatives are called *second order differential equation.* For example many equations arising from Newton 2nd law of motion;

$$\text{mass} \times \text{acceleration} = \text{force}$$

are second order differential equations. Now if the force is gravity then

$$m\frac{d^2y}{dt^2} = -mg.$$

If the force comes from a spring then

$$m\frac{d^2y}{dt^2} = -ky.$$

This is Hookes law which says that the restoring force is proportional to $y$.

Sometimes you can treat these as first order equations for velocity and solve for velocity as a function of time.

eg. if $m\frac{d^2y}{dt^2} = -mg$ and we let $v = \frac{dy}{dt}$, then $\frac{dv}{dt} = -g$ is a first order equation for $v(t)$.
But it only works if the right hand side is not a function of $y$.