4. Differential Equations

4.1 Motion of Projectiles

Gravity in one dimension.

Let's start with an apple falling under gravity.

If y is the distance vertically up then

$$\frac{md^2y}{dt^2} = -mg \Rightarrow \frac{d^2y}{dt^2} = -g.$$

Let $v = \dfrac{dy}{dt}$ then

$$\frac{dv}{dt} = -g \quad \Rightarrow \quad \int dv = -\int g\,dt.$$

So $v = -gt + c$, where c is the integration constant.

If $v(0) = v_0 \Rightarrow c = v_0$. So $v = -gt + v_0$. Now we have $\dfrac{dy}{dt} = -gt + v_0$.

Integrate again

$$\int dy = \int (-gt + v_0)dt \quad \Rightarrow \quad y = -g\frac{t^2}{2} + v_0 t + c'.$$

Now if $y(0) = y_0 \Rightarrow c' = y_0$. So $y = -\dfrac{1}{2}gt^2 + v_0 t + y_0$.

Projectile motion.

Now suppose you are firing apples over the neighbours fence with the aim of hitting a target 15 meters off.

If the projectile has position $(x(t), y(t))$ and your firing device is positioned at the origin then $x(0) = 0$ and $y(0) = 0$ then we now have two equations of motion;

$$\frac{d^2y}{dt^2} = -g \quad \text{and} \quad \frac{d^2x}{dt^2} = 0.$$

Suppose further that the initial horizontal velocity is $u_0 = \dfrac{dx}{dt}(0)$ and vertically $v_0 = \dfrac{dy}{dt}(0)$. Now solving $\dfrac{d^2y}{dt^2} = -g$ gives $y = -\dfrac{1}{2}gt^2 + v_0 t$ as before.

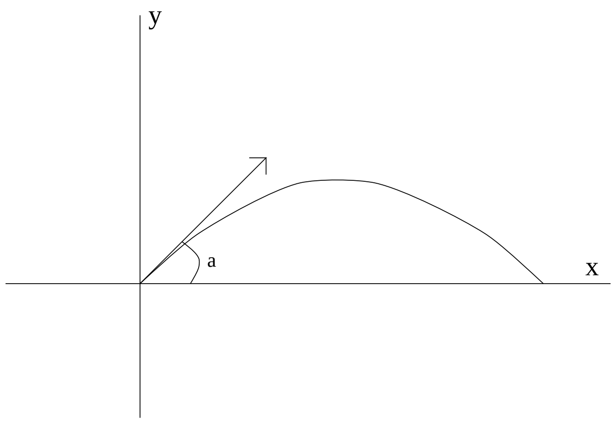
To solve $\dfrac{d^2x}{dt^2} = 0$ we let $u = \dfrac{dx}{dt}$

$$\Rightarrow \frac{du}{dt} = 0 \quad \Rightarrow \quad u = u_0 \Rightarrow \frac{dx}{dt} = u_0$$

$$\Rightarrow \quad \int dx = \int u_0 dt \Rightarrow x = u_0 t + c \text{ but } c = 0 \text{ because } x_0 = 0.$$

Therefore $y = -\frac{1}{2}gt^2 + v_0 t$ and $x = u_0 t$.

Example. If your firing device only fires at 5m/s what angle should it fire at to hit your target 15m away?



Suppose the angle is α. Then $u_0 = 5\cos\alpha$ and $v_0 = 5\sin\alpha$.

Now the projectile hits the ground when $y = 0$. So

$$-\frac{1}{2}gt^2 + v_0 t = 0 \Rightarrow t(v_0 - \frac{1}{2}gt) = 0 \Rightarrow t = 0 \text{ and } t = \frac{2v_0}{g}.$$

At that point $x = u_0 t = \frac{2u_0 v_0}{g} = \frac{50\cos\alpha\sin\alpha}{g}$.

So if $15 = \frac{50\cos\alpha\sin\alpha}{g} \quad \Rightarrow \quad \sin 2\alpha = \frac{15}{25}g = \frac{3}{5}g \quad (2\cos\alpha\sin\alpha = \sin 2\alpha)$.

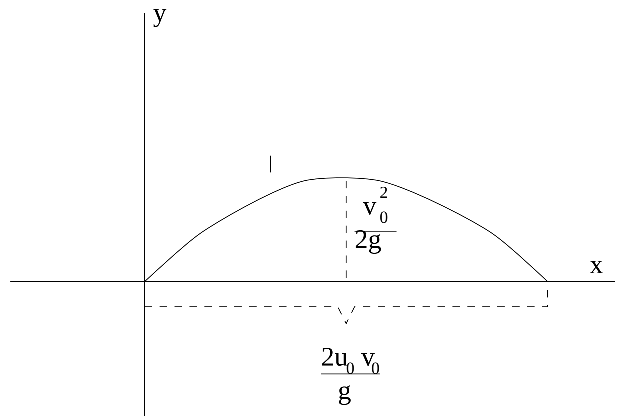
Solving for α we obtain $\alpha = \frac{1}{2}(2k\pi + \arcsin(\frac{3g}{5}))$ or $\alpha = \frac{1}{2}(2k\pi + \pi - \arcsin(\frac{3g}{5}))$.

Example. What sort of curve does a projectile trace out in space?

To make the equations simpler we will take $x_0 = 0$ and $y_0 = 0$. Then $y = -\frac{1}{2}gt^2 + v_0 t$ and $x = u_0 t$. From $x = u_0 t$ we have $\frac{x}{u_0} = t$. So

$$\begin{aligned}
\Rightarrow y &= -\frac{gx^2}{2u_0^2} + \frac{v_0}{u_0}x \\
&= -\frac{g}{2u_0^2}\left(x^2 - \frac{2u_0 v_0}{g}x\right) \text{ parabola} \\
&= -\frac{g}{2u_0^2}\left(\left(x - \frac{u_0 v_0}{g}\right)^2 - \frac{u_0^2 v_0^2}{g^2}\right) \\
&= -\frac{g}{2u_0^2}\left(x - \frac{u_0 v_0}{g}\right)^2 + \frac{v_0^2}{2g}.
\end{aligned}$$

This is a parabola with max height at $x = \dfrac{u_0 v_0}{g}$. The max hight is $y = \dfrac{v_0^2}{2g} = h_{\max}$ and range is $x = \dfrac{2u_0 v_0}{g}$.



Note if $x_0 \neq 0$ or $y_0 \neq 0$ we simply replace x by $x - x_0$ and y by $y - y_0$.