4. Differential Equations

4.2 Slope Fields and Equilibrium Solutions Slope Fields

A first order differential equation

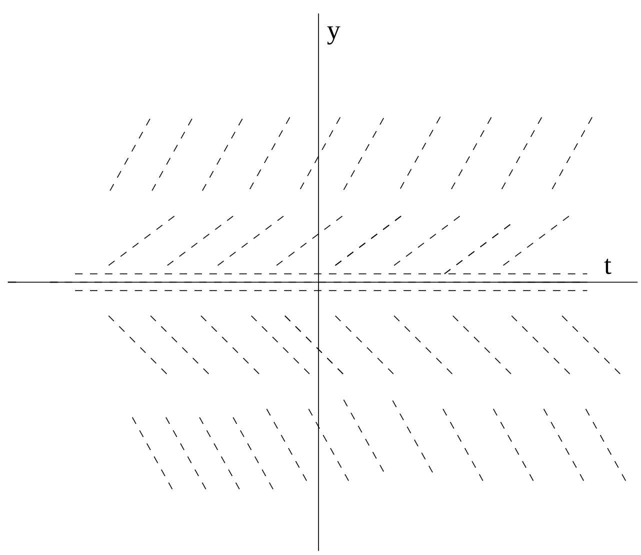
$$\frac{dy}{dt} = f(y, t)$$

actually tells you the slope of the solution curve at any point.

For example - if $\frac{dy}{dt} = 2y$ then

at	$y = 0$	the slope is $0 (= 2 \times 0)$
at	$y = 1$	the slope is $2 (= 2 \times 1)$
	$y = 2$	the slope is 4
	$y = -1$	the slope is -2
	$y = -2$	the slope is -4

From this you can build up a picture, called a *slope field*, of the solution curves without even solving the differential equation. But it gets much better the more points you use, so it is wise to use matlab (quiver), or maple (dfield plot).



From the slope field of $\frac{dy}{dt} = 2y$ we can see that $y = 0$ is a solution curve. It is a constant solution - or equilibrium solution.

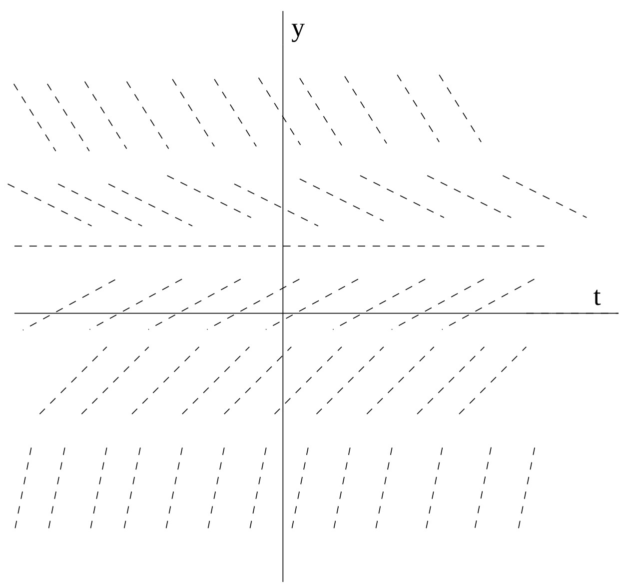
Equilibrium Solutions are constant solutions to an ODE $\frac{dy}{dt} = f(y,t)$. The graph is a horizontal line, so the slope is zero.

$$\Rightarrow \frac{dy_e}{dt} = 0 \Rightarrow f(y_e,\, t) = 0.$$

Example. Find the equilibrium solutions of $\frac{dy}{dt} = -3(y-1)$ and sketch the slope field. Equilibrium solutions are constant solutions. So $-3(y-1) = 0$. This implies $y = 1$.

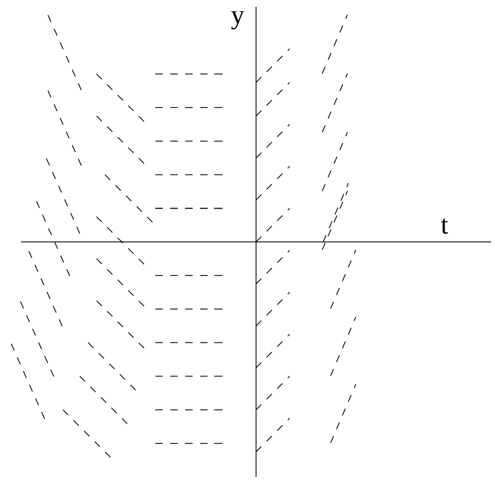
$$\begin{array}{lcll} \text{At } y = -1 & \Rightarrow & \frac{dy}{dt} = 6 & \text{The slope is 6} \\ \text{At } y = 0 & \Rightarrow & \frac{dy}{dt} = 3 & \text{The slope is 3} \\ \text{At } y = \frac{1}{2} & & \frac{dy}{dt} = \frac{3}{2} & \\ \text{At } y = 1 & & \frac{dy}{dt} = 0 & \\ \text{At } y = 2 & & \frac{dy}{dt} = -3 & \\ \text{At } y = 3 & & \frac{dy}{dt} = -6 & \end{array}$$

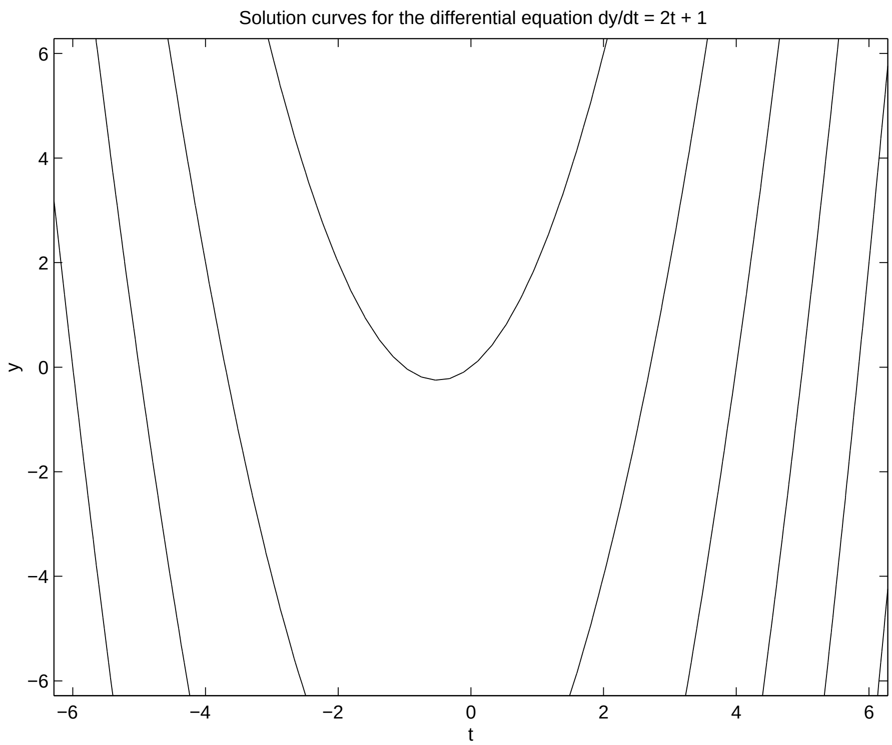
All solutions tend to $y = 1$. If $y < 1$ then y is increasing to $y = 1$. If $y > 1$ then y is decreasing to $y = 1$.



Example. Find the equilibrium solutions of $\frac{dy}{dt} = 2t + 1$ and sketch the slope field.
To obtain equilibrium solutions we set $\frac{dy}{dt} = 0$; or $2t + 1 = 0$. This gives no constant value for y. So there is no equilibrium solutions for this equation.

$$
\begin{aligned}
t &= -1 & \Rightarrow \quad \frac{dy}{dt} &= -1 \\
t &= -\frac{1}{2} & \frac{dy}{dt} &= 0 \\
t &= 0 & \frac{dy}{dt} &= 1 \\
t &= \frac{1}{2} & \frac{dy}{dt} &= 2
\end{aligned}
$$



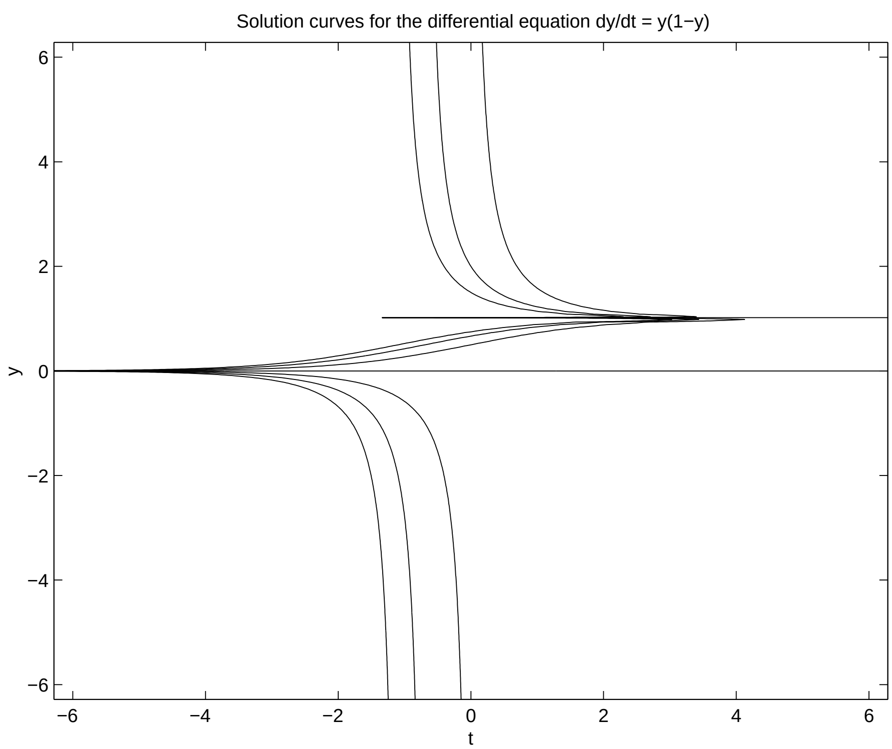


Example. Find the Equilibrium Solutions of $\frac{dy}{dt} = y(1-y)$ and sketch the slope field.

Equilibrium solutions $y(1-y) = 0 \Rightarrow y = 0$ or $y = 1$.

At

$$
\begin{aligned}
y &= \frac{1}{4} && \text{slope} \quad \frac{dy}{dt} = \frac{1}{4}\cdot\frac{3}{4} = \frac{3}{16}\\
y &= \frac{1}{2} && \text{slope} \quad = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\\
y &= \frac{3}{4} && \quad ” \quad = \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16}\\
y &= 1\frac{1}{2} && \quad ” \quad = \frac{3}{2}\cdot(-\frac{1}{2}) = \frac{-3}{4}\\
y &= -\frac{1}{2} && \quad ” \quad = -\frac{1}{2}\cdot\frac{3}{2} = \frac{-3}{4}\\
y &= 2 && \quad = 2(-1) = -2\\
y &= -1 && \quad = -1(2) = -2.
\end{aligned}
$$



If you start with $y > 0$ then solutions tend to $y = 1$.

If initially $y < 0$ then $y \to -\infty$.

If initially $y = 0$ or $y = 1$ then y stays zero or one.

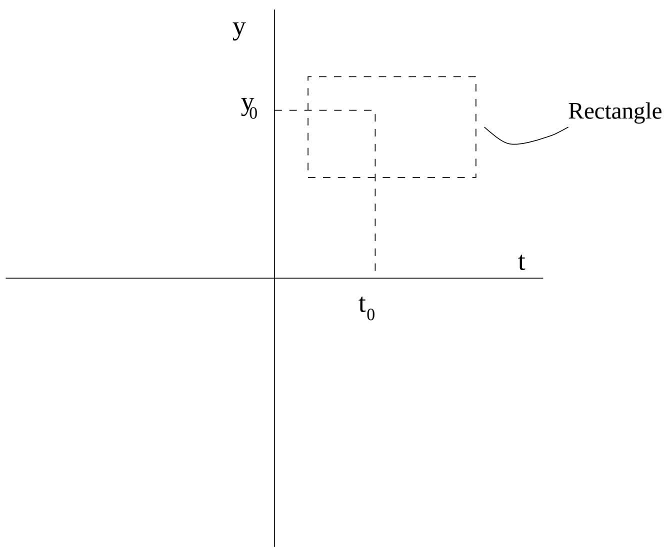
Existence and Uniqueness of Solutions

Take an initial value problem

$$\frac{dy}{dt} = f(y, t), \quad y(t_0) = y_0.$$

Then if $f(y, t)$ is "smooth" in some rectangle about (y_0, t_0) there is (ie exists) a unique solution $y = y_1(t)$ at least in some neighbourhood of (y_0, t_0).

This means that equilibrium solutions cannot be crossed (in fact no solutions can cross) and that they divide up the (y, t) solution space.



So for instance solutions to $\dfrac{dy}{dt} = -3(y-1)$ that start with $y(t_0) < 1$ stay less than 1.