

## 4. Differential Equations

### 4.3 Euler's Method for Solving DE's Numerically

Euler's method uses tangent lines as approximations to solution curves. The tangent line to a solution curve of  $\frac{dy}{dt} = f(y, t)$  at  $(y_0, t_0)$  can be constructed exactly, without solving for  $y(t)$  because the differential equation itself tells you the slope of the solution curve at  $(y_0, t_0)$ . The slope is  $f(y_0, t_0)$ .

So the tangent line at  $(y_0, t_0)$  is

$$y = y_0 + f(y_0, t_0)(t - t_0)$$

which approximates the curve for

$$t \simeq t_0 \text{ or } \Delta t = t - t_0 \quad \text{small.}$$

So  $y(t_0 + \Delta t) \simeq y_0 + f(y_0, t_0)\Delta t$ . We will call it  $y_1 = y_0 + f(y_0, t_0)\Delta t$ , the approximate value of  $y$  after one small step in time.

Now imagine a family of solution curves to the differential equation. We can calculate an approximate value for  $y$  at some later time by taking lots of small steps. At each step we will use the tangent line to the solution pathing through that point.

Let

$$t_1 = t_0 + \Delta t, \quad t_2 = t_1 + \Delta t, \quad t_3 = t_2 + \Delta t, \quad \dots, \quad t_n = t_{n-1} + \Delta t.$$

Take the first step to  $(y_1, t_1)$

$$y_1 = y_0 + f(y_0, t_0)\Delta t$$

Now use your approximate value of  $y(t_1) \simeq y_1$ , to take the next step.

$$y_2 = y_1 + f(y_1, t_1)\Delta t \quad \text{then } y_2 \simeq y(t_2).$$

In general  $y_{n+1} = y_n + f(y_n, t_n)\Delta t$ .

**Example.** Use Euler's Method to find an approximate solution to the initial value problem

$$\frac{dy}{dt} = 2t, \quad y(0) = 0$$

for  $y(0.6)$  with  $\Delta t = 0.2$ .

We note that  $f(y_n, t_n) = 2t_n$ . Now make the following table:

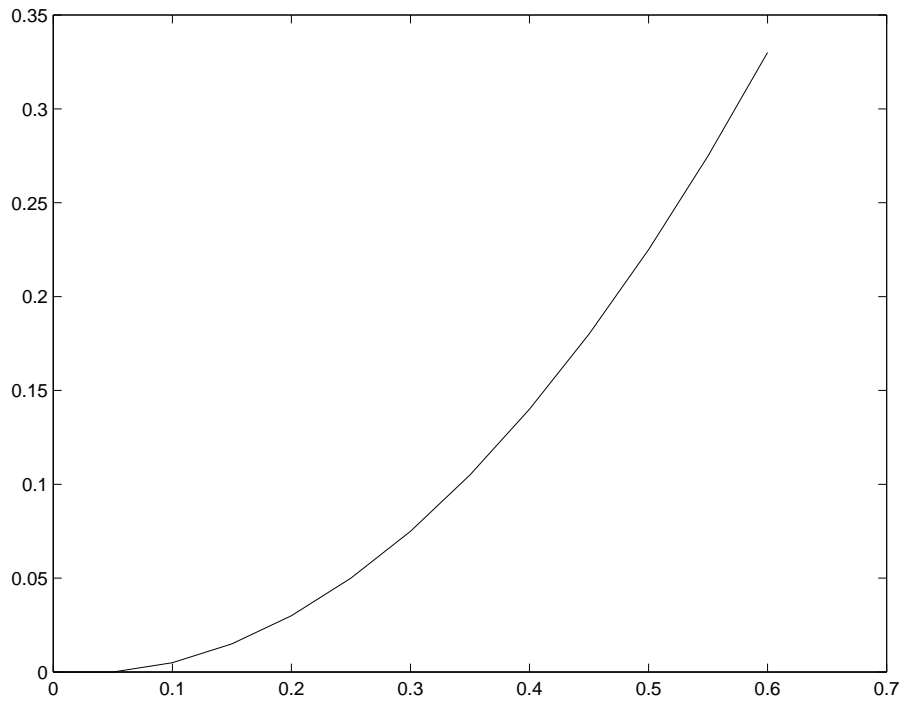
$t_n$	$f(y_n, t_n)$	$y_{n+1}$
0	$2 \times 0 = 0$	$0 + 0 \times 0.2 = 0$
0.2	$2 \times 0.2 = 0.4$	$0 + 0.4 \times 0.2 = 0.08$
0.4	$2 \times 0.4 = 0.8$	$0.08 + 0.8 \times 0.2 = 0.24$
0.6	$2 \times 0.6 = 1.2$	$0.24 + 1.2 \times 0.2 = 0.48$

So  $y_3 = 0.48 \simeq y(0.6)$ . The actual value is 0.36!

Of course the method is only accurate if you make  $\Delta t$  small and then the number of steps you have to take is large. So use matlab. Define a  $t$  vector and an initial  $y$  value, then use the for command to iterate. In the example  $\Delta t = 0.05$  and 12 steps have been taken to take  $t$  to 0.6.

```
t = (0 : 0.05 : 0.6);
y(1) = 0;
for i = 1 : 12
    y(i + 1) = y(i) + 2 * t(i) * 0.05;
end
y(13)
plot (t, y, ' -')
```

Using Matlab we find  $y(0.6) = 0.33$  which is much much better than the previous estimate value for  $y(0.6)$ .



**Example.** Consider the differential equation

$$\frac{dy}{dx} = \sin(xy), \quad \text{with initial condition } y(0) = 1.$$

Estimate  $y(2)$ , using Euler's method with step size  $\Delta x = 0.01$ .

```
x = (0 : 0.01 : 2);
```

```
y(1) = 1;
```

```
for i = 1 : 200
```

```
    y(i + 1) = y(i) + sin(x(i) * y(i)) * 0.01;
```

```
end
```

```
y(201)
```

```
plot (x, y, '-')
```

Using Matlab we find  $y(2) = 1.8243$ .

