4. Differential Equations

4.4 Solving Separable Equations

To solve $\frac{dy}{dt} = t$ we write the equation in differential form

$$\begin{aligned} dy &= t\,dt \\ \text{and integrate} \quad \int dy &= \int t\,dt \\ \Longrightarrow y &= \frac{t^2}{2} + c. \end{aligned}$$

We can do essentially the same thing with a whole class of more general equations. These are called *separable equations*. For example if $\frac{dy}{dt} = 2y$ then $\frac{dy}{y} = 2dt$. Now integrate

$$\begin{aligned} \int \frac{dy}{y} &= \int 2\,dt \\ \ln|y| &= 2t + c \\ |y| &= e^{2t+c} = e^c e^{2t} \\ y &= Ae^{2t}, \quad \text{where } A = \pm e^c. \end{aligned}$$

In fact any equation that can be put in the form $\frac{dy}{dt} = \frac{f(t)}{g(y)}$ can be separated, i.e.

$$g(y)dy = f(t)dt.$$

Then we integrate.

Example. Solve the initial value problem $\frac{dy}{dt} = te^y$ with $y(0) = 0$

$$\begin{aligned} \frac{dy}{e^y} &= tdt \\ \Longrightarrow \quad \int e^{-y} dy &= \int tdt \\ \Longrightarrow \quad -e^{-y} &= \frac{t^2}{2} + c \\ \Longrightarrow y &= -\ln\left|-c - \frac{t^2}{2}\right| \\ y(0) = 0 \Longrightarrow c &= -1. \quad \text{So } y = \ln\left|1 - \frac{t^2}{2}\right|. \end{aligned}$$

Quiz: Which of the following equations are separable?

(a) $\dfrac{dy}{dt} = y(1-y)$

(b) $\dfrac{dy}{dt} = e^{t+y}$

(c) $\dfrac{dy}{dt} = e^{(t+y)^2}$

(d) $\dfrac{dy}{dt} = \dfrac{ty+y}{t^2}$

(e) $\dfrac{dy}{dt} = ty + y^2.$

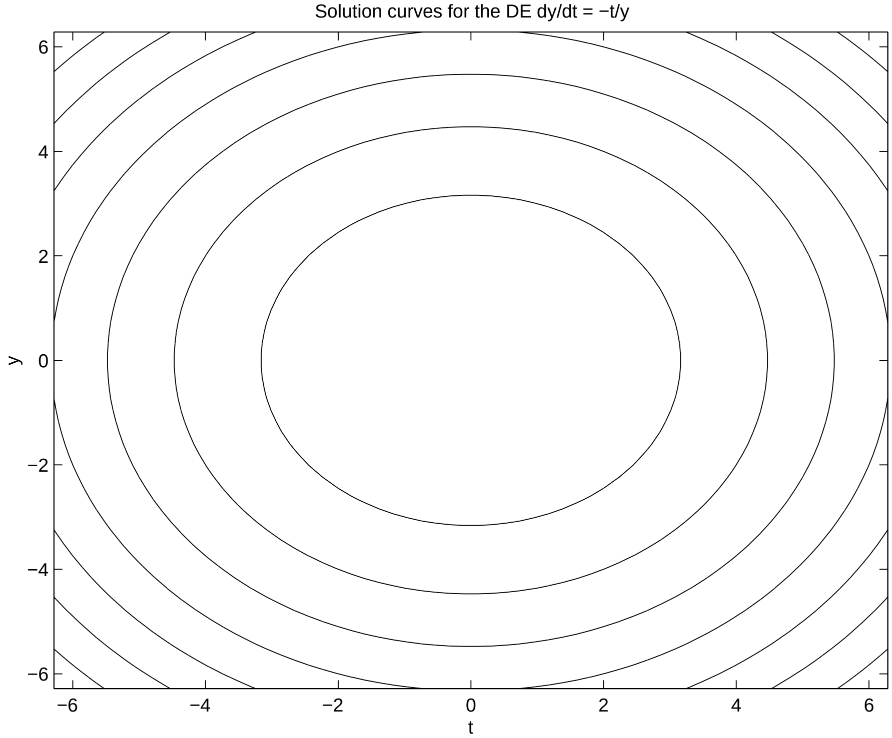
Example. Solve $\dfrac{dy}{dt} = \dfrac{-t}{y}$ and sketch the solution curves in $(t,\, y)$ space

Separate: $\quad ydy = -tdt$

Integrate: $\quad \dfrac{y^2}{2} = -\dfrac{t^2}{2} + c$

$\Rightarrow \; y^2 + t^2 = 2c \quad$ circles with radius $\sqrt{2c}$.

Solution curves for the DE dy/dt = −t/y



Example. Solve the following initial value problem

$$\frac{dy}{dx} = \frac{\sin x}{y}, \qquad y(0) = 1$$

$$\Rightarrow \int y\,dy = \int \sin x\,dx \Rightarrow \frac{y^2}{2} = -\cos x + c$$

$$\Rightarrow y = \pm\sqrt{2c - 2\cos x}$$

$$y(0) = 1 \Rightarrow 1 = \pm\sqrt{2c - 2}.$$

This means we must take the + square root and $c = \frac{3}{2}$.

Example. Solve the logistic equation

$$\frac{dy}{dt} = y(1-y) \quad \text{with} \quad y(0) = y_0.$$

Separate

$$\int \frac{dy}{y(1-y)} = \int dt.$$

Now use partial fractions

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} \quad \Rightarrow \quad 1 = A(1-y) + By \quad \text{for all } y$$

$$\Rightarrow \quad 1 = A + (B - A)y.$$

So $A = 1$ and $B - A = 0 \Rightarrow B = 1$.
So $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$ and the equation becomes

$$
\begin{aligned}
& \int \frac{dy}{y} + \int \frac{dy}{1-y} = \int dt \\
\Rightarrow \quad & \ln|y| - \ln|1-y| = t + c \\
\Rightarrow \quad & \ln\left|\frac{y}{1-y}\right| = t + c \\
\Rightarrow \quad & \left|\frac{y}{1-y}\right| = e^c e^t \\
\Rightarrow \quad & \frac{y}{1-y} = Ae^t, \quad \text{where} \quad A = \pm e^c \\
\Rightarrow \quad & y = Ae^t(1-y) \\
\Rightarrow \quad & y(1 + Ae^t) = Ae^t \\
\Rightarrow \quad & y = \frac{Ae^t}{1 + Ae^t}.
\end{aligned}
$$

Now we use the initial condition to find A.

Since

$$\frac{y}{1-y} = Ae^t \quad \Rightarrow \quad \frac{y_0}{1-y_0} = A.$$

So

$$y = \frac{y_0 e^t}{(1-y_0) + y_0 e^t}$$

after multiplying top and bottom by $(1-y_0)$.

Note as $t \to +\infty$ we obtain $y \to 1$.