4. Differential Equations

4.5 Applications, heat equation, population growth.

Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and the surrounding temperature. So if T is the temperature of the body and S is the surrounding temperature then

$$\text{Rate of change of temperature } = \frac{dT}{dt} = -\lambda(T-S),$$

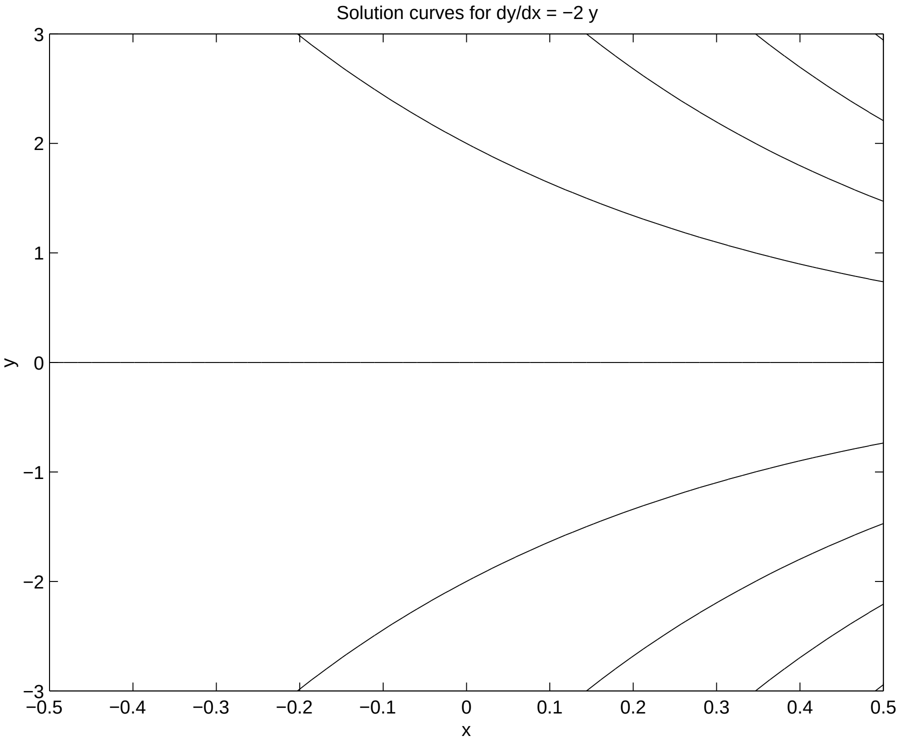
where λ is the constant of proportionality.

Example. Suppose the surrounding temperature is 0°C and at $t = 0$ your cup of coffee which is initially at T_0 cools at a rate of 2 C/s. How long will it take to cool to T_1?

First find the solution curves

$$\begin{aligned} \frac{dT}{dt} = -2T \quad &\Rightarrow \quad \int \frac{dT}{T} = \int -2dt \\ &\Rightarrow \quad \ln|T| = -2t + c \\ &\Rightarrow \quad T = Ae^{-2t} \end{aligned}$$

where $A = T(0)$.



What do the solution curves look like if $S \neq 0$? In that case

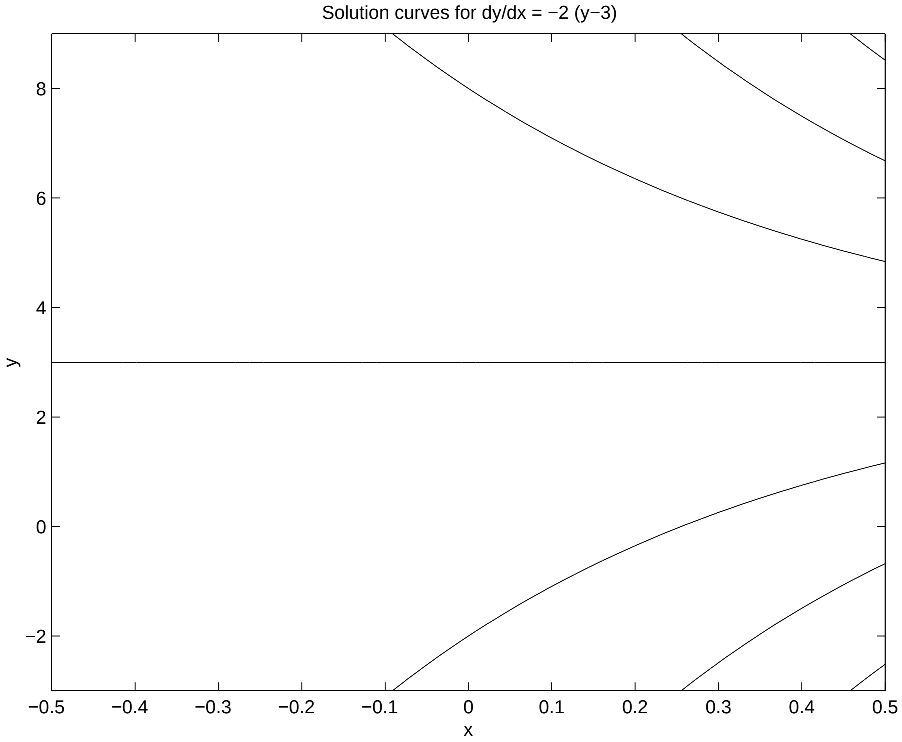
$$\frac{dT}{(T-S)} = -\lambda dt$$

$$\Rightarrow \ln|T-S| = -\lambda t + c$$

$$\Rightarrow T - S = Ae^{-\lambda t}$$

$$\Rightarrow T = S + Ae^{-\lambda t}$$

$T = S$ is a *stable* equilibrium solution because near by solutions tend towards.



Often in a real problem λ is not given and you have to work it out from the information you are given.

Example. **Police Report**

Body discovered at 11 pm.

Doctor arrived at 11.30 pm., body temp 94.6°F.

Doctor took the temperature again at 12.30 and found the body temperature to be 93.4°F.

It was noted that the temperature of the room was 70°F.

Can you estimate the time of death? Let T be the temperature of the body then

$$\frac{dT}{dt} = -\lambda(T - S) \quad \text{and} \quad S = 70.$$

Let t be the time in hours after 11.30.

$$\int \frac{dT}{T - 70} = \int -\lambda dt \Rightarrow T = 70 + Ae^{-\lambda t}.$$

We are given that

$$\begin{aligned} T(0) = 94.6 \quad &\Rightarrow \quad 94.6 = 70 + A \\ &\Rightarrow \quad A = 24.6. \end{aligned}$$

So $T = 70 + 24.6e^{-\lambda t}$.

Now one hour later $T(1) = 93.4 = 70 + 24.6e^{-\lambda}$

$$\Rightarrow e^{-\lambda} = \frac{23.4}{24.6} \text{ or } \lambda = -\ln\left(\frac{23.4}{24.6}\right) = 0.0500.$$

So $T = 70 + 24.6e^{-0.05t}$.

Now body temperature, at least of a living body, is about 97.4°F. So we need to know at what time t_D, the body temperature was 97.4°F.

$$\begin{aligned} T(t_D) = 97.4 &= 70 + 24.6e^{-0.05t_D} \\ \Rightarrow -0.05t_D &= \ln\left(\frac{27.4}{24.6}\right) \\ \Rightarrow t_D &= -2.1559 = -(2 \text{ hours } 9 \text{ mins}). \end{aligned}$$

So time of death is 11.30-2.09 = **9.21 pm**.

Population Growth, Exponential Growth.

Without inhibiting factors the rate at which a population grows is proportional to the existing population. So if P is the population at time t then

$$\frac{dP}{dt} = rP,$$

where r is then the growth rate per head per year.

For the population of the world $r \approx 0.02$. So

$$\frac{dP}{dt} = 0.02P \Rightarrow P = Ae^{0.02t}, \qquad \text{where } A = P(0).$$

Example. How long will it take to double the worlds population? What is interesting about this question is that you can answer it without knowing the present population. Lets suppose the population is $P(0)$ then we want to know t such that $P(t) = 2P(0)$

$$\Rightarrow 2P(0) = P(0)e^{0.02t}$$

$P(0)$ cancels to the doubling time $t = \dfrac{\ln 2}{0.02} \approx 35$ years.
In general if $\dfrac{dP}{dt} = rP \Rightarrow P = P(0)e^{rt}$ doubling time is $\left(\dfrac{\ln 2}{r}\right)$.

Example. If in two years a rabbit population trebles, how long does it take for the population to increase 9 fold? Let $N(t)$ be the rabbit population. Then assuming exponential growth

$$\frac{dN}{dt} = rN \Rightarrow N = N(0)e^{rt}.$$

We do not know r, but we can work it out. Since

$$\begin{aligned} N(2) = 3N(0) &\Rightarrow 3N(0) = N(0)e^{2r} \\ &\Rightarrow e^{2r} = 3 \Rightarrow r = \frac{\ln 3}{2}. \end{aligned}$$

Since

$$e^{rt} = e^{(\frac{\ln 3}{2})t} = e^{(t/2)\ln 3} = e^{\ln 3^{t/2}} = 3^{t/2}$$

if

$$\begin{aligned} N(t) = 9N(0) &\Rightarrow 9N(0) = N(0)e^{rt} \\ &\Rightarrow 9 = 3^{t/2} \Rightarrow \frac{t}{2} = 2 \Rightarrow t = 4. \end{aligned}$$

So it takes 4 years to increase the population 9 fold.

Nonlinear Model of Population Growth

In most animal populations some factor, such as over crowding or limited food supply, inhibits growth. To model this we introduce a nonlinear term which only becomes important if the population is large.

$$\frac{dP}{dt} = rP - \frac{rP^2}{L} = rP\left(1 - \frac{P}{L}\right) \text{ where } L \text{ is large.}$$

This is called the logistic equation.

There are two equilibrium solutions: $P = 0$ and $P = L$.

If $0 < P < L$ then $\frac{dP}{dt} > 0$ ie P increases towards $P = L$ and if $L < P$ then $\frac{dP}{dt} < 0$ ie P decreases.

If we let $y = \frac{P}{L}$ and $\tau = rt$ the equation becomes $\frac{dy}{d\tau} = y(1 - y)$ which has the solution

$$y = \frac{y_0 e^{\tau}}{(1 - y_0) + y_0 e^{\tau}}.$$

So

$$P = \frac{LP_0 e^{rt}}{(L - P_0) + P_0 e^{rt}}.$$

For $0 < P_0 < L$ we see that $P \to L$ as $t \to +\infty$ and $P \to 0$ as $t \to -\infty$.

Also for $L < P_0$ we see that $P_0 \to L$ as $t \to +\infty$. (L is called the Carrying Capacity.)

Example. The Pacific halibut fishery has been modelled by

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{L}\right) \quad \text{(logistic equation)}$$

where $P(t)$ is the mass of fish in kg.

Time is measured in years and $r = 0.71$ per year. The carrying capacity $L = 8 \times 10^7$kg.

If $P(0) = 2 \times 10^7$ how long will it take to reach 4×10^7?

We know that $P = \frac{LP(0)e^{rt}}{(L - P(0)) + P(0)e^{rt}}$.

So we need to solve for t if

$$\begin{aligned}
4 \times 10^7 &= \frac{8 \times 10^7 \times 2 \times 10^7 e^{rt}}{(8-2) \times 10^7 + 2 \times 10^7 e^{rt}} \\
\frac{1}{2} &= \frac{1}{3e^{-rt} + 1} \\
\Rightarrow e^{-rt} &= \frac{1}{3} \\
t &= \frac{\ln 3}{r} \simeq 1.55 \text{ years.}
\end{aligned}$$

Example. Change the above model to take account of harvesting at a rate proportional to the existing population. If E is the constant of proportionality what restriction must be set on E for the fish population *not* to die out?

Including harvesting

$$\frac{dP}{dt} = rP(1-\frac{P}{L}) - EP = P(r - E - r\frac{P}{L}).$$

Now the equilibrium solutions are $P = 0$ and $P = L(1-\frac{E}{r})$, a reduced carrying capacity. If $E > r$ the population tends to zero, ie it dies out. So we require $E < 0.71$.

But what value of E will ensure a maximum sustainable harvest? In the long term P tends to the carrying capacity, so the population is $P \simeq L(1-\frac{E}{r})$ and this yields $Y(E) = EP \simeq EL(1-\frac{E}{r})$.

Now the maximum of E occurs where $\frac{dY}{dE} = 0$

$$\frac{dY}{dE} = L(1-\frac{2E}{r}) = 0 \Rightarrow E = \frac{r}{2}$$

and $Y_{\max}(\frac{r}{2}) = \frac{rL}{2}(\frac{1}{2}) = \frac{rL}{4} = \frac{0.71 \times 8 \times 10^7}{4} = 1.42 \times 10^7$.

Solving DE's numerically

Matlab has much more sophisticated numerical methods than Eulers. Matlabs *ode45* uses a Runge Kutter, fourth order integration technique. To use it you need to write two M-files. The first one defines the differential equation. Say call it logistic.m:

function $\quad dy = \text{logistic}(t, y)$

$dy = y * (1 - y)$.

Use a second M-file to integrate and plot your solution;

$y0 = 0.1$;

$t$final = 6;

[t, y] = ode45 ('logistic', $t$final, $y0$);

plot (t, y).