4. Differential Equations

4.6 Solving Linear First Order Equations

By linear we mean equations that are linear in the dependent variable, y, and its derivative; $\frac{dy}{dt}$. So they are equations of the form

$$\frac{dy}{dt} = a(t)y + b(t).$$

If a and b are constants the equation is separable. (It has the solution $y = -\frac{b}{a} + Ae^{at}$.)
If $b(t) = 0$ the equation is separable:

$$\begin{aligned} \int \frac{dy}{y} = \int a(t)dt \quad &\Rightarrow \quad \ln|y| = \int a(t)dt + c \\ &\Rightarrow \quad y = Ae^{\int a(t)dt}, \quad \text{if} \quad y \geq 0 \end{aligned}$$

which gives us an idea!

$$\frac{d}{dt}\left(ye^{-\int a(t)dt}\right) = 0 \quad \text{if} \quad b(t) = 0$$

because

$$\frac{d}{dt}\left(ye^{-\int a(t)dt}\right) = \left(\frac{dy}{dt} - a(t)y\right)e^{-\int a(t)dt}.$$

But if $b(t) \neq 0$

$$\begin{aligned} \frac{d}{dt}\left(ye^{-\int a(t)dt}\right) &= e^{-\int a(t)dt}\left(\frac{dy}{dt} - a(t)y\right) \\ &= b(t)e^{-\int a(t)dt} \end{aligned}$$

which in itself is an equation we can solve.
Lets try an example $\frac{dy}{dt} = y + t$ say.
Here $e^{-\int a(t)dt} = e^{-t}$. It is called the *integrating factor* and

$$\frac{d}{dt}(ye^{-t}) = \left(\frac{dy}{dt} - y\right)e^{-t} = te^{-t}.$$

This is an equation we can solve

$$\begin{aligned} \int d(ye^{-t}) &= \int te^{-t}dt && \text{Use integration by parts} \\ \Rightarrow ye^{-t} &= -te^{-t} + \int e^{-t}dt && \int udv = uv - \int vdu \\ &= -te^{-t} - e^{-t} + c && \text{here let } u = t,\ dv = e^{-t}dt \\ \Rightarrow y &= -t - 1 + e^{t}c && \Rightarrow du = dt,\ v = -e^{-t} \end{aligned}$$

(Check it works: $\dfrac{dy}{dt} = -1 + ce^t$ and $y + t = -1 + ce^t$.)

Method for solving Linear First Order DEs

1. Put the equation in the form
$$\frac{dy}{dt} = a(t)y + b(t).$$
2. Solve for the integrating factor
$$\mu(t) = e^{-\int a(t)dt}.$$
Then
$$\frac{d}{dt}(\mu(t)y) = \mu(t)b(t).$$
3. Integrate this equation
$$\mu(t)y = \int \mu(t)b(t)dt + c$$
and solve it for y.

Example. Solve the initial value problem
$$x^2\frac{dy}{dx} + xy = 1 \quad \text{with} \quad y(1) = 2$$
$$\Rightarrow \frac{dy}{dx} = -(\frac{1}{x})y + \frac{1}{x^2}.$$

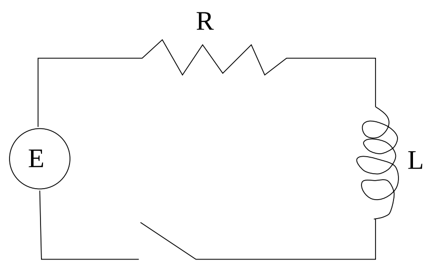
Integrating factor
$$\mu(x) = e^{-\int -\frac{1}{x}dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x \Rightarrow \frac{d}{dx}(xy) = x(\frac{1}{x^2}) = \frac{1}{x}.$$

(Not a bad idea to check at this point: $\dfrac{d}{dx}(xy) = x\dfrac{dy}{dx} + y = \dfrac{1}{x}$.)

Integrate
$$\begin{aligned} xy &= \int \frac{dx}{x} + c \\ &= \ln|x| + c \\ \Rightarrow y &= \frac{\ln x}{x} + \frac{c}{x}, \quad \text{if } x > 0 \\ y(1) &= \frac{\ln 1}{1} + c = c, \quad \text{so} \quad c = 2 \\ &\Rightarrow \; y(x) = \frac{\ln x + 2}{x} \text{ is the solution.} \end{aligned}$$

Electric Circuits

The equation for the current in a circuit with a resistor and an inductor turns out to be linear. To get the equation consider the change in voltage around the circuit. The voltage drop due to a resistor is RI where R is the resistance (in Ohms) and I is the current (in amps).



The voltage drop due to the inductor is $L\frac{dI}{dt}$ where L is the inductance (henries).
Now Kirchhoff's Law tells us that the sum of the voltage drops is equal to the supplied voltage $E(t)$ (volts).
So $E(t) = RI + L\frac{dI}{dt}$ or $L\frac{dI}{dt} + RI = E(t)$.

Example. Suppose in an L, R circuit a battery supplies a constant voltage of 80V, the inductance is 2H and the resistance is 10Ω.

a) Find $I(t)$

b) What is the current after 0.1 sec if $I(0) = 0$?

The equation becomes

$$2\frac{dI}{dt} + 10I = 80 \Rightarrow \frac{dI}{dt} = -5I + 40.$$

Integrating factor

$$\mu(t) = e^{-\int -5dt} = e^{5t} \Rightarrow \frac{d}{dt}(e^{5t}I) = 40e^{5t}.$$

Integrate

$$e^{5t}I = \frac{40}{5}e^{5t} + c \Rightarrow \quad I = 8 + ce^{-5t}.$$

Now if $t = 0$ then $I == 8 + c$. So if $I(0) = 0$ then $c = -8$ and $I(t) = 8 - 8e^{-5t}$.

Now $I(0.1) = 8 - 8e^{-0.5} = 3.1478$ amps.

But what if a generator supplies a variable voltage, so that $E(t) = 80 \sin 3t$.

$$\Rightarrow 2\frac{dI}{dt} + 10I = 80 \sin 3t \Rightarrow \frac{dI}{dt} = -5I + 40 \sin 3t.$$

The Integrating factor is as before $\mu(t) = e^{5t}$

$$\Rightarrow \frac{d}{dt}(e^{5t}I) = 40e^{5t} \sin 3t.$$

Integrate! Which means we need to know how to calculate

$$J = \int e^{5t} \sin 3t \, dt.$$

This can be done by integration by parts twice!

$$\int u dv = uv - \int v du.$$

Let $u = e^{5t}$ and $dv = \sin 3t \, dt \Rightarrow du = 5e^{5t} dt$ and $v = \dfrac{-\cos 3t}{3}$.

$$J = \int e^{5t} \sin 3t \, dt = -\frac{e^{5t} \cos 3t}{3} + \frac{5}{3} \int e^{5t} \cos 3t \, dt.$$

Now let $u = e^{5t}$ and $dv = \cos 3t \, dt \Rightarrow du = 5e^{5t}$ and $v = \dfrac{\sin 3t}{3}$.

$$\begin{aligned} J &= -\frac{e^{5t} \cos 3t}{3} + \frac{5}{3}\left(\frac{e^{5t} \sin 3t}{3} - \frac{5}{3} \int e^{5t} \sin 3t \, dt\right) \\ &= -\frac{e^{5t}}{9}(3 \cos 3t - 5 \sin 3t) - \frac{25}{9} J \\ \Rightarrow (1 + \frac{25}{9})J &= -\frac{e^{5t}}{9}(3 \cos 3t - 5 \sin 3t) \\ \Rightarrow J &= -\frac{e^{5t}}{34}(3 \cos 3t - 5 \sin 3t). \end{aligned}$$

Now back to the DE

$$\begin{aligned} e^{5t} I &= -\frac{40}{34} e^{5t}(3 \cos 3t - 5 \sin 3t) + c \\ \Rightarrow \quad I(t) &= -\frac{40}{34}(3 \cos 3t - 5 \sin 3t) + ce^{-5t}. \end{aligned}$$

Now if $I(0) = 0 \Rightarrow c = \dfrac{120}{34}$.

So $I(t) = -\dfrac{40}{34}(3 \cos 3t - 5 \sin 3t) + \dfrac{60e^{-5t}}{17}$. The exponential term soon dies.