4. Differential Equations

4.7 Linear Homogeneous Second order Differential Equations with Constant Coefficients.

Second order differential equations involve 2nd order derivatives, $\frac{d^2y}{dt^2}$, of the dependent variable, and often first order derivatives, $\frac{dy}{dt}$, as well. If they are linear then they are linear in $\frac{d^2y}{dt^2}$, $\frac{dy}{dt}$ and y. So they have the form

$$\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t).$$

If $f(t) = 0$ they are said to be *homogeneous* and if the coefficients b and c are constants the solutions are relatively simple.

Consider first the solution of a *first order* homogeneous constant coefficient equation.

$$\frac{dy}{dt} = ay \Rightarrow y = y_0 e^{at}, \text{ which is exponential.}$$

Can you guess any exponential solutions to

$$\frac{d^2y}{dt^2} - 4y = 0?$$

One way to find exponential solutions is simply to assume $y = e^{rt}$ and substitute y into the equation:

$$y = e^{rt} \Rightarrow \frac{dy}{dt} = re^{rt} \Rightarrow \frac{d^2y}{dt^2} = r^2 e^{rt}.$$

Now substitute into

$$\begin{aligned} \frac{d^2y}{dt^2} = 4y \quad &\Rightarrow \quad r^2 e^{rt} = 4e^{rt} \\ &\Rightarrow \quad r^2 = 4 \Rightarrow r = \pm 2. \end{aligned}$$

So $y_1 = e^{2t}$ and $y_2 = e^{-2t}$ are both solutions.

Finding exponential solutions

Constant coefficient homogeneous linear ODE's $\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$ have at least one exponential type solution. To find it let $y = e^{rt}$. Then differentiate and substitute in.

$$y = e^{rt} \Rightarrow \frac{dy}{dt} = re^{rt} \Rightarrow \frac{d^2y}{dt^2} = r^2 e^{rt}$$

$$r^2 e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Cancel the $e^{rt}(\neq 0) \Rightarrow r^2 + br + c = 0$. This is called Auxillary Equation. We then solve it for r.

Example. Find the exponential solutions of

$$\ddot{y} - 3\dot{y} + 2y = 0.$$

Let $y = e^{rt} \Rightarrow \dfrac{dy}{dt} = re^{rt}$ and $\dfrac{d^2y}{dt^2} = r^2e^{rt}$.
Substitute in $r^2e^{rt} - 3re^{rt} + 2e^{rt} = 0$. Since $e^{rt} \neq 0$ it follows that

$$\begin{aligned} & r^2 - 3r + 2 = 0 \qquad \text{is the Auxillary Equation} \\ & (r-2)(r-1) = 0 \\ \Rightarrow\ & r = 2 \quad \text{or} \quad r = 1. \end{aligned}$$

So $y_1 = e^{2t}$ and $y_2 = e^t$ are both solutions to the DE.
Because the equation is linear and homogeneous if $y_1 = e^{2t}$ is a solution, then so is c_1e^{2t} for any constant c_1. Try it by simply substituting in.
Similarily c_2e^t is a solution for any constant c_2. Perhaps even more suprisingly the sum of any two solutions is a solution.
Let $y = y_1 + y_2$ where y_1 and y_2 are solutions to the DE. So $\ddot{y}_1 + b\dot{y}_1 + cy_1 = 0$ and $\ddot{y}_2 + b\dot{y}_2 + cy_2 = 0$. Now

$$(\ddot{y}_1 + \ddot{y}_2) + b(\dot{y}_1 + \dot{y}_2) + c(y_1 + y_2) = \ddot{y}_1 + b\dot{y}_1 + cy_1 + \ddot{y}_2 + b\dot{y}_2 + cy_2 = 0.$$

So $y_1 + y_2$ is also a solution.
This means that $y = c_1e^{2t} + c_2e^t$ must be a solution to $\ddot{y} - 3\dot{y} + 2y = 0$ for any constants c_1 and c_2. In fact it is the most general solution.

Theorem. If $y_1(t)$ and $y_2(t)$ are solutions of

$$\frac{d^2y}{dt^2} + a(t)\frac{dy}{dt} + b(t)y = 0 \qquad (*)$$

and if $y_2(t)$ is not simply a multiple of $y_1(t)$ then the general solution to (*) is

$$y = c_1y_1(t) + c_2y_2(t).$$

So that any solution is of this form for some constants c_1 and c_2.

For example the general solution to $\ddot{y} - 3\dot{y} + 2y = 0$ is $y = c_1 e^{2t} + c_2 e^t$.

Example. Find the general solution to

$$\ddot{y} + 4\dot{y} - 5y = 0.$$

Let $y = e^{rt} \Rightarrow \dfrac{dy}{dt} = re^{rt} \Rightarrow \dfrac{d^2y}{dt^2} = r^2 e^{rt}$. Substitute into $\ddot{y} + 4\dot{y} - 5y = 0$ to obtain $r^2 e^{rt} + 4re^{rt} - 5e^{rt} = 0$.

$$\Rightarrow \quad r^2 + 4r - 5 = 0 \qquad \text{(Auxillary Equation)}$$
$$(r+5)(r-1) = 0.$$

So e^{-5t} and e^t are solutions and one is not simply a multiple of the other. Therefore, the general solution is $y = c_1 e^{-5t} + c_2 e^t$.

But it is not always as easy as this.

Consider the DE $\ddot{y} + b\dot{y} + cy = 0$. Let $y = e^{rt} \Rightarrow \dot{y} = re^{rt} \Rightarrow \ddot{y} = r^2 e^{rt}$ and substitute into $\ddot{y} + b\dot{y} + cy = 0$ to obtain

$$e^{rt}(r^2 + br + c) = 0$$
$$\Rightarrow r^2 + br + c = 0 \qquad \text{(Auxillary Equation).}$$

The auxillary equation may have

Case 1: Two real different roots r_1 and r_2.

This happens if $b^2 > 4c$. In this case the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

But it could also have (case 2) equal roots $r = \dfrac{-b}{2}$ if $b^2 = 4c$ or (case 3) complex roots if $b^2 < 4c$.

Case 2: Equal roots $r = -\dfrac{b}{2}$. This happens if $b^2 = 4c$. So $y_1 = e^{-\frac{b}{2}t}$ is a solution and since the equation is linear $c_1 e^{-\frac{b}{2}t}$ is a solution for any constant c_1. But we have no second solution. So we will try a method called *variation of parameters*. The idea is to let $v(t)$ be a function of time and look for a solution of the form $y = v(t)e^{-\frac{b}{2}t}$.

Let's take an example. Suppose we want to find the general solution to $\ddot{y} + 2\dot{y} + y = 0$. The auxillary equation is $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$. So one solution is e^{-t}.

Now let $y = v(t)e^{-t} \Rightarrow \dfrac{dy}{dt} = \dfrac{dv}{dt}e^{-t} - ve^{-t}$ and $\dfrac{d^2y}{dt^2} = \left(\dfrac{d^2v}{dt^2} - \dfrac{2dv}{dt} + v\right)e^{-t}$. Now substitute into the equation.

$$(\ddot{v} - 2\dot{v} + v)e^{-t} + 2(\dot{v} - v)e^{-t} + ve^{-t} = 0$$

$$\Rightarrow \ddot{v} = 0 \Rightarrow \dot{v} = c_2 \quad \text{and} \quad v(t) = c_2 t + c_1.$$

So $y = (c_2 t + c_1)e^{-t} = c_1 e^{-t} + c_2 t e^{-t}$ which is the general solution.

The second solution is te^{-t}. In general the second solution is t times the exponential.

So if $b^2 = 4c$ (equal roots) then $r = -\frac{b}{2}$ and the general solution is $y = c_1 e^{-\frac{b}{2}t} + c_2 t e^{-\frac{b}{2}t}$.

Example. Solve the following initial value problem.

$$4\ddot{y} - 4\dot{y} + y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 3.$$

Let $y = e^{rt} \Rightarrow \dot{y} = re^{rt}$ and $\ddot{y} = r^2 e^{rt}$ and substitute into $4\ddot{y} - 4\dot{y} + y = 0$ to obtain

$$(4r^2 - 4r + 1)e^{rt} = 0 \Rightarrow (2r-1)^2 = 0 \Rightarrow \text{ (equal roots) } r = \frac{1}{2}.$$

So the general solution is

$$y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}.$$

Now $y(0) = 1 \Rightarrow c_1 = 1$ and $\dot{y} = \dfrac{1}{2}c_1 e^{\frac{1}{2}t} + c_2(\dfrac{t}{2} + 1)e^{\frac{1}{2}t}$. So

$$\dot{y}(0) = \frac{1}{2}c_1 + c_2 = 3 \Rightarrow c_2 = \frac{5}{2}.$$

Therefore, the solution is $y = e^{\frac{1}{2}t} + \dfrac{5}{2}te^{\frac{1}{2}t}$.

Case 3: Complex Roots ($b^2 < 4c$)

Since the coefficients b and c of the auxillary equation $r^2 + br + c = 0$ are real it follows that the complex roots

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \alpha \pm i\beta$$

appear as $\alpha + i\beta$ and its complex conjugate $\alpha - i\beta$.

In the complex plane the general solution is $y = D_1 e^{(\alpha + i\beta)t} + D_2 e^{(\alpha - i\beta)t}$ where D_1 and D_2 are complex constants.

But in real applications we need real solutions which means we need to be able to take the real part of y.

Recall complex numbers if

$$\begin{aligned} z_1 &= a + ib \quad , \quad z_2 = c + id \\ z_1 z_2 &= (a+ib)(c+id) = ac - bd + i(bc + ad), \quad i^2 = -1 \\ e^{ia} &= \cos a + i \sin a \quad \text{(Euler's Formula).} \end{aligned}$$

So $e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i(\beta t)} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$ and $e^{(\alpha-i\beta t)} = e^{\alpha t}(\cos\beta t - i\sin\beta t)$.

Now if $D_1 = D_{1r} + iD_{1i}$ and $D_2 = D_{2r} + iD_{2i}$

$$\begin{aligned} D_1 e^{(\alpha+i\beta)t} + D_2 e^{(\alpha-i\beta)t} &= e^{\alpha t}\left((D_{1r}+D_{2r})\cos\beta t - (D_{1i}+D_{2i})\sin\beta t \right. \\ &\quad \left. + i(D_{1i}+D_{2i})\cos\beta t + i(D_{1r}-D_{2r})\sin\beta t.\right) \end{aligned}$$

So the real part has the form

$$y_{\text{real}} = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$$

where c_1 and c_2 are real constants.

Therefore, if $r = \alpha \pm i\beta$ then the general solution is

$$y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t.$$

Example. Find the general solution to

$$\ddot{y} + 9y = 0.$$

Let $y = e^{rt} \Rightarrow \dot{y} = re^{rt}$ and $\ddot{y} = r^2 e^{rt}$. Now substitute into $\ddot{y} + 9y = 0$ to obtain

$$(r^2 + 0)e^{rt} = 0 \Rightarrow r = \pm 3i.$$

So general solution is $y = c_1 \cos 3t + c_2 \sin 3t$.

Example. Solve the following initial value problem

$$\ddot{y} - 4\dot{y} + 5y = 0, \quad y(0) = 1\,, \quad \dot{y}(0) = 0.$$

Let $y = e^{rt} \Rightarrow \dot{y} = re^{rt}$ and $\ddot{y} = r^2 e^{rt}$. So

$$\Rightarrow e^{rt}(r^2 - 4r + 5) = 0 \Rightarrow r = 2 \pm i$$

$$\Rightarrow y = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t$$

$$y(0) = c_1 \Rightarrow c_1 = 1.$$

Now

$$\dot{y} = c_1(2e^{2t} \cos t - e^{2t} \sin t) + c_2(2e^{2t} \sin t + e^{2t} \cos t)$$

$$\dot{y}(0) = c_1(2) + c_2(1) = 0 \Rightarrow c_2 = -2$$

$$\text{So} \qquad y = e^{2t} \cos t - 2e^{2t} \sin t.$$

Summary

To solve $\ddot{y} + b\dot{y} + cy = 0$.

Let $y = e^{rt}$ then r is given by the auxillary equation $r^2 + br + c = 0$.

Case 1 $b^2 > 4c \Rightarrow$ two real distinct roots r_1 and r_2.

General solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

Case 2 $b^2 = 4c \Rightarrow$ equal roots $r = -\dfrac{b}{2}$.

General solution is $y = c_1 e^{-\frac{b}{2}t} + c_2 t e^{-\frac{b}{2}t}$.

Case 3 $b^2 < 4c \Rightarrow$ complex roots $r = \alpha \pm i\beta$.

General solution is $y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$.

Solve the following equations

(1) $\ddot{y} + y = 0$

(2) $\ddot{y} - 6\dot{y} - 27y = 0$

(3) $\ddot{y} + 6\dot{y} + 9y = 0$

(4) $\ddot{y} - y = 0$

(5) $\ddot{y} + 2\dot{y} + 26y = 0$