4. Differential Equations

4.8 Simple Harmonic Motion and damped oscillations

Mass on a spring

If the force of gravity exactly equals the force exerted by the spring the mass is in the equilibrium position. Displace it a bit, pull it down or up, and it will oscillate. If there is no damping it would oscillate forever simple harmonic motion. A small amount of damping and the oscillations slowly die away. If the damping is really large it may never oscillate.

Equations of motion for Simple Harmonic motion - no damping

Hooke's Law tells us the force exerted by a spring and Newton's second law of motion relates the force to acceleration;

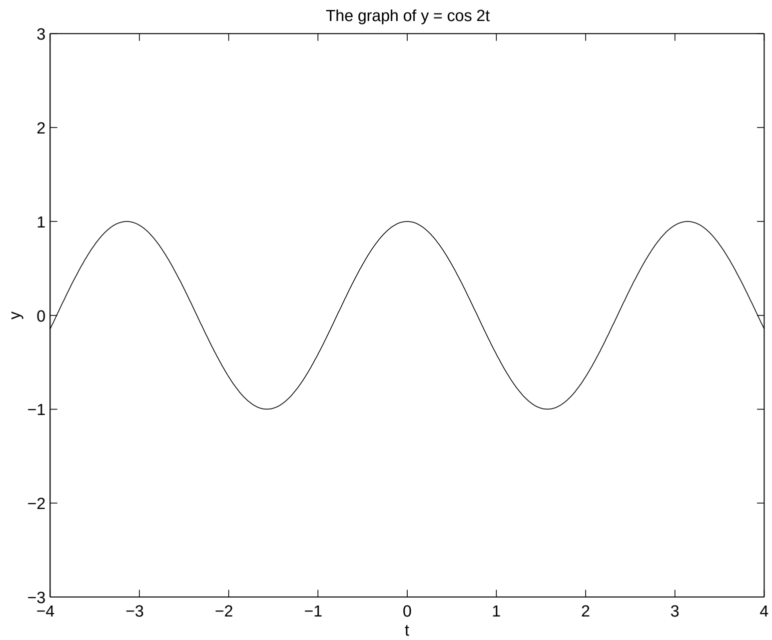
$$\text{mass} \times \text{acceration} = \text{force}.$$

Now if y is the displacement from equilibrium, Hooke's law tells us that the force is proportional to y, so that without damping

$$m\frac{d^2y}{dt^2} = -ky \quad \Rightarrow \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0,$$

where k is the spring constant.

Example. Solve the equation of motion for a spring mass system with mass 1 kg and spring constant 4.
Putting for k and m in above DE we obtain $\frac{d^2y}{dt^2} + 4y = 0$. Now let $y = e^{rt}$ then the auxillary equation is $r^2 + 4 = 0$. This implies $r = \pm 2i$. So $y = c_1 \cos 2t + c_2 \sin 2t$ (oscillatory with period $\frac{2\pi}{2} = \pi$).
But what do the solutions look like?
If $y(0) = 1$ and $\frac{dy}{dt}(0) = 0$ ie pull the mass down and let go. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $\frac{dy}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t$ with $\frac{dy}{dt}(0) = 0 \Rightarrow c_2 = 0$. So that $y = \cos 2t$.



But if $y(0) = 1$ and $\frac{dy}{dt}(0) = 2$ ie pull the mass down and give the mass a kick as you let go. Then

$$y(0) = 1 \Rightarrow c_1 = 1$$
$$\frac{dy}{dt}(0) = 2 \Rightarrow 2c_2 = 2 \Rightarrow c_2 = 1.$$

So that $y = \cos 2t + \sin 2t$ which still oscillates with period π but amongst other things the oscillations have a larger amplitude.

To understand this we need the *addition formula* for sine.

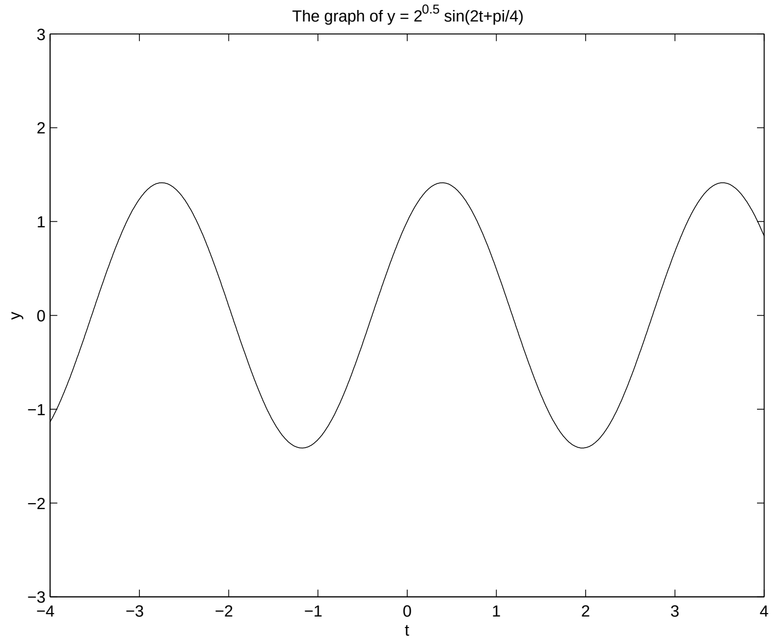
$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

which can be proved geometrically. Now we prove $y = \sqrt{2}\sin(2t + \pi/4)$.

$$\begin{aligned} \sqrt{2}\sin(2t + \pi/4) &= \sqrt{2}\sin\ 2t\cos(\pi/4) + \sqrt{2}\cos\ 2t\sin(\pi/4) \\ &= \sqrt{2}\sin\ 2t(\sqrt{2}/2) + \sqrt{2}\cos\ 2t(\sqrt{2}/2) \\ &= \sin\ 2t + \cos\ 2t. \end{aligned}$$

So $y = \sqrt{2}\sin(2t + \pi/4)$ which is a sine wave with amplitude $\sqrt{2}$, period π and phase shift $\pi/4$.

We can now sketch the solution, noticing that at $t = -\frac{\pi}{8}$ we have $y = 0$.



In general if

$$y = c_1 \cos \omega t + c_2 \sin \omega t$$

and we let $\tan \phi = \dfrac{c_1}{c_2}$ and $A = \sqrt{c_1^2 + c_2^2}$ then $A \cos \phi = c_2$ and $A \sin \phi = c_1$. So

$$\begin{aligned} A \sin(\omega t + \phi) &= A \sin \omega t \cos \phi + A \cos \omega t \sin \phi \\ &= c_2 \sin \omega t + c_1 \cos \omega t \\ &= y. \end{aligned}$$

(Oscillations with ampalitude $A = \sqrt{c_1^2 + c_2^2}$, period $\dfrac{2\pi}{\omega}$ and phase shift arctan $\dfrac{c_1}{c_2}$.)

Example. A spring with mass 9 kg and spring constant $4N$ is pulled down 1 m and given a kick of $-\dfrac{1}{2}$ m/s. Solve for its position as a function of time and sketch your solution.

The equation of motion is

$$9\frac{d^2y}{dt^2} + 4y = 0.$$

Let $y = e^{rt} \Rightarrow 9r^2 + 4 = 0$ is the auxillary equation.

So $r = \pm\dfrac{2}{3}i$ and $y = c_1 \cos \dfrac{2}{3}t + c_2 \sin \dfrac{2}{3}t$ (oscillations with period $\dfrac{2\pi}{2}.3 = 3\pi$).

Now $y(0) = 1 \Rightarrow c_1 = 1$ and

$$\frac{dy}{dt} = -\frac{2}{3}c_1 \sin \frac{2}{3}t + \frac{2}{3}c_2 \cos \frac{2}{3}t.$$

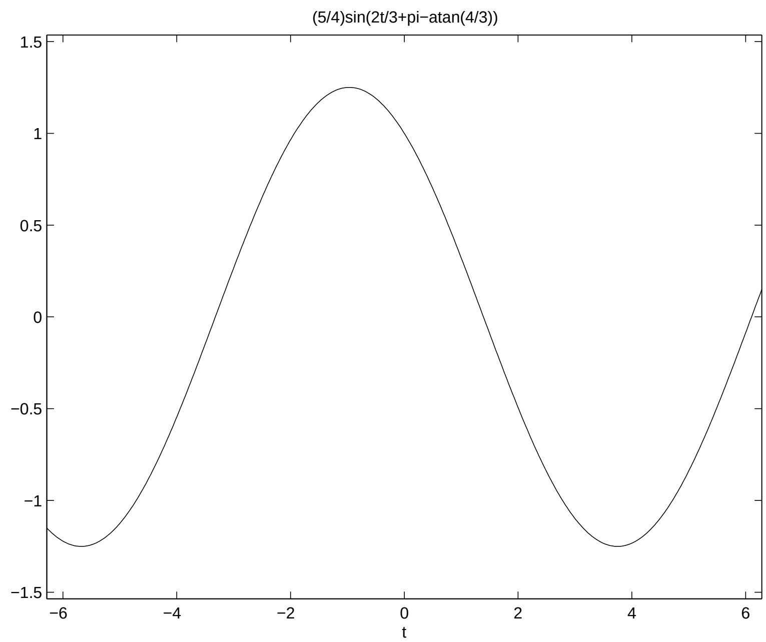
So

$$\frac{dy}{dt}(0) = -\frac{1}{2} \Rightarrow c_2 = -\frac{3}{4}$$
$$y = \cos\frac{2}{3}t - \frac{3}{4}\sin\frac{2}{3}t.$$

Now $\tan\phi = -\frac{1}{\frac{3}{4}} = -\frac{4}{3} \quad \Rightarrow \quad \text{phase shift } = \phi = \arctan\frac{-4}{3},$

and amplitude $= \sqrt{1 + \frac{(-3)^2}{4^2}} = \frac{5}{4}.$

So $y = \frac{5}{4}\sin\left(\frac{2}{3}t + \phi\right)$ where $\phi = \arctan\frac{-4}{3}.$



Damped Oscillations.

Damping proportional to velocity also hinders the motion. So from Newtons 2nd law

$$m\frac{d^2y}{dt^2} = -a\frac{dy}{dt} - ky$$
$$\Rightarrow m\frac{d^2y}{dt^2} + a\frac{dy}{dt} + ky = 0$$

where a is the damping coefficient.

Example with small damping Let $a = 0.2$, $k = 4$ N and $m = 1$ kg then

$$\Rightarrow \frac{d^2y}{dt^2} + 0.2\frac{dy}{dt} + 4y = 0.$$

Now if $y = e^{rt}$ then the auxillary equation is $r^2 + 0.2r + 4 = 0$. Solving for r we obtain

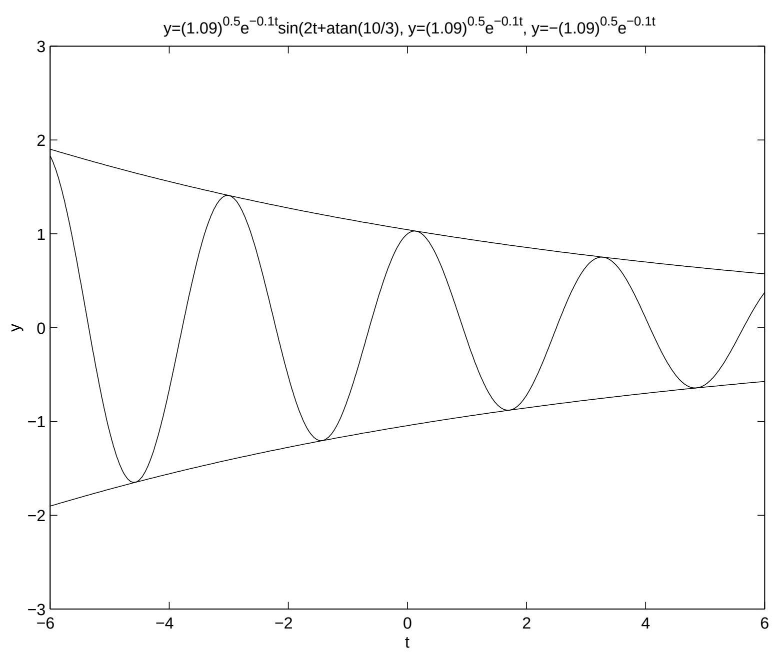
$$r = -0.1 \pm \sqrt{0.01 - 4} \simeq -0.1 \pm 2i$$
$$y = c_1 e^{-0.1t} \cos 2t + c_2 e^{-0.1t} \sin 2t$$
$$= e^{-0.1t} A \sin(2t + \phi)$$

Oscillations period π, phase shift ϕ but now the amplitude $Ae^{-0.1t}$ decays slowly in time. Solving for A and ϕ now gets messy.

Example. $a = 0.2$, $k = 4$ N and $m = 1$ kg, as before, and $y(0) = 1$ m, $\dot{y}(0) = \frac{1}{2}$ m/s.

$$y(0) = 1 \Rightarrow c_1 = 1$$
$$\frac{dy}{dt} = c_1(-0.1e^{-0.1t} \cos 2t - 2e^{-0.1t} \sin 2t) + c_2(-0.1e^{-0.1} \sin 2t + 2e^{-0.1t} \cos 2t)$$
$$\dot{y}(0) = -0.1c_1 + 2c_2 = \frac{1}{2} \Rightarrow c_2 = 0.3$$
$$y = e^{-0.1t}(\cos 2t + 0.3 \sin 2t) = e^{-0.1t}\sqrt{1 + 0.09} \sin(2t + \phi),$$

where $\phi = \arctan \frac{1}{0.3} = \arctan \frac{10}{3}$.



What happens if damping is increased?

In general

$$m\frac{d^2y}{dt^2} + a\frac{dy}{dt} + ky = 0$$

has auxillary equation $mr^2 + ar + k = 0$

$$\Rightarrow r = \frac{-a \pm \sqrt{a^2 - 4km}}{2m}$$

which has complex roots if $a^2 < 4km$ and real roots, both negative, if $a^2 > 4km$.

- If $a = 2\sqrt{km}$ the system is *critically damped*.
- If $a < 2\sqrt{km}$ the solutions are damped oscillations.
- But if $a > 2\sqrt{km}$ the solutions do not oscillate - over damping.

Example. Solve for the motion of a mass 1 kg on a spring, spring constant 4 if the damping coefficient is 5 and the mass is displaced 1m from equilibrium and released.

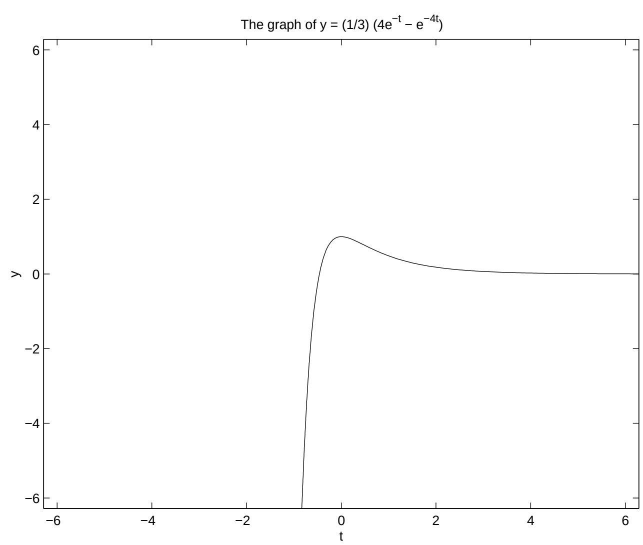
$$1\ddot{y} + 5\dot{y} + 4y = 0 \text{ with } y(0) = 1\,,\ \dot{y}(0) = 0.$$

Auxillary equation is $r^2 + 5r + 4 = 0$; that is $(r+4)(r+1) = 0$.

$$y = c_1e^{-4t} + c_2e^{-t} \quad \Rightarrow \quad \dot{y} = -4c_1e^{-4t} - c_2e^{-t}$$

$$y(0) = 1 \Rightarrow c_1 + c_2 = 1 \quad \text{and} \quad \dot{y}(0) = 0 \Rightarrow 4c_1 + c_2 = 0 \Rightarrow c_2 = -4c_1$$

$$\Rightarrow c_1 = -\frac{1}{3} \text{ and } c_2 = \frac{4}{3} \quad \Rightarrow \quad y = \frac{(4e^{-t} - e^{-4t})}{3}.$$



Example. If a mass of 1kg is suspended from a spring, spring constant 4, what value of the damping constant a implies critical damping?

$$a = 2\sqrt{km} = 2\sqrt{4} = 4.$$

Then $r = \dfrac{-a}{2m} = -2$ and the solutions are $y = c_1 e^{-2t} + c_2 t e^{-2t}$.

Summary of motion of a spring:

$$m\frac{d^2y}{dt^2} + a\frac{dy}{dt} + ky = 0$$

where m is the mass, a is the damping coefficient and k is the spring constant.

Auxillary equation $mr^2 + ar + k = 0$ has complex roots if $a^2 < 4km$.

(1) $a = 0$ No damping - Pure oscillatory motion: $r = \pm\sqrt{\dfrac{k}{m}}i$

$$y = c_1 \cos\sqrt{\frac{k}{m}}t + c_2 \sin\sqrt{\frac{k}{m}}t.$$

(2) $a < 2\sqrt{km}$ Damped oscillations: $r = \alpha \pm i\beta$ where $\alpha = -\dfrac{a}{2m} < 0$

$$y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t.$$

(3) $a = 2\sqrt{km}$ Critical Damping: Equal roots $r = -\dfrac{a}{2m}$

$$y = c_1 e^{-\frac{a}{2m}t} + c_2 t e^{-\frac{a}{2m}t}.$$

(4) $a > 2\sqrt{km}$ Over Damping: $r_1 < 0$ and $r_2 < 0$

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

A Pendulum also executes simple harmonic motion and damped oscillations.

Suppose a bob moves in an arc of a circle radius ℓ. The distance moved by the bob is $\ell\theta$.

Newton's 2nd law tells us

$$\begin{aligned} &\text{mass} \times \frac{d^2}{dt^2}(\ell\theta) = \text{force in the direction of motion} \\ &\Rightarrow m\ell\ddot{\theta} = -mg\sin\theta \\ &\Rightarrow \ell\ddot{\theta} = -g\sin\theta \text{ which is a non linear equation.} \end{aligned}$$

But if θ is small $\sin\theta \simeq \theta$.

So for θ small

$$\ell\ddot{\theta} + g\theta = 0.$$

Let $\theta = e^{rt} \Rightarrow r^2 + \frac{g}{\ell} = 0$ is the auxillary equation.

$$r = \pm\sqrt{\frac{g}{\ell}}i$$

$$\theta = c_1 \cos\sqrt{\frac{g}{\ell}}t + c_2 \sin\sqrt{\frac{g}{\ell}}t.$$

Oscillations period is $2\pi\sqrt{\frac{\ell}{g}}$.

What is the effect of damping?

As before damping will be proportional to velocity $\Rightarrow \ell\ddot{\theta} + R\dot{\theta} + g\theta = 0$ where R is damping coefficient. Then auxillary equation is $\ell r^2 + Rr + g = 0$

$$\Rightarrow r = \frac{-R}{2\ell} \pm \frac{\sqrt{R^2 - 4\ell g}}{2\ell}.$$

If

$R^2 < 4\ell g$	r is complex $\Rightarrow$ damped oscillations
$R = 2\sqrt{\ell g}$	critical damping
$R > 2\sqrt{\ell g}$	solutions are overdamped.