5. Parametrisation of Curves and Line Integrals

5.1 Parametrisation of Curves

Circle

Imagine you are in a racing car on a circular track, radius 1 km. Someone in a plane will see you trace out the circle $x^2 + y^2 = 1$. But to describe your actual motion we need $x(t)$ and $y(t)$.

One parametrisation of the circle would be $x(t) = \cos\omega t$ and $y = \sin\omega t$, which starts at (1, 0).

Check $(x(t), y(t))$ lies on the circle $x^2 + y^2 = 1$:

$$x^2 + y^2 = \cos^2\omega t + \sin^2\omega t = 1.$$

Then ω controls the speed you are travelling at.

The circumference of the circle is 2π and the time it takes to go all the way round is $t = \dfrac{2\pi}{\omega}$. *So the speed is* ω.

But this is not the only parametrisation of the circle. Say you started at (0, 1) and traversed the circle clockwise. Then

$$\begin{cases} x(t) = \cos\omega\left(\dfrac{\pi}{2\omega} - t\right) & \Rightarrow \quad x(t) = \cos\left(\dfrac{\pi}{2} - \omega t\right) \\ y(t) = \sin\omega\left(\dfrac{\pi}{2\omega} - t\right) & \Rightarrow \quad y(t) = \sin\left(\dfrac{\pi}{2} - \omega t\right). \end{cases}$$

You can use addition formulae or graphs to simplify $x(t)$ and $y(t)$.

Recall

$$\begin{cases} \sin(a+b) = \sin a\cos b + \cos a\sin b \\ \cos(a+b) = \cos a\cos b - \sin a\sin b \end{cases}$$

So

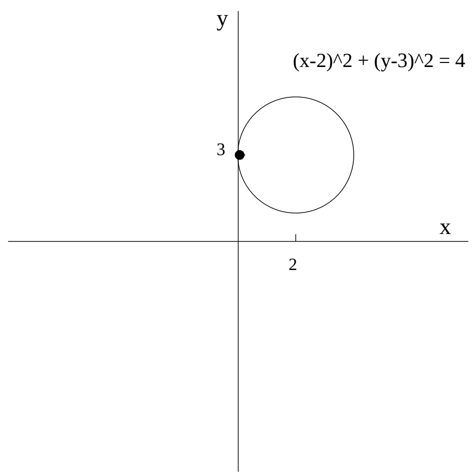
$$x(t) = \cos\frac{\pi}{2}\cos\omega t + \sin\frac{\pi}{2}\sin\omega t = \sin\omega t$$
$$y(t) = \sin\frac{\pi}{2}\cos\omega t - \cos\frac{\pi}{2}\sin\omega t = \cos\omega t.$$

In general if you start at (x_0, y_0) on a circle with radius a then

$$x(t) = a\cos(\theta \pm \omega t) \quad \text{and} \quad y(t) = a\sin(\theta \pm \omega t),$$

where $\theta =$ the polar angle $= \arctan\left(\dfrac{y_0}{x_0}\right)$, and $+$ is for counter clockwise rotation and $-$ is for clockwise rotation.

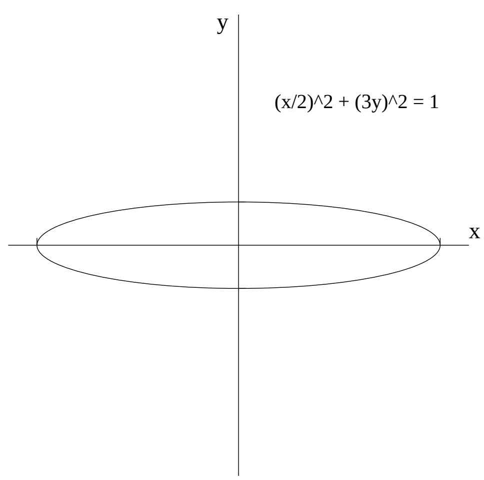
Example. Parametrise a circle centred $(2, 3)$ radius 2, starting at $(0, 3)$ and traversed in a counter clockwise direction.



The equation of the circle is $(x-2)^2 + (y-3)^2 = 4$. So

$$\begin{cases} x - 2 = 2\cos(\pi + t) & \Rightarrow \quad x = 2 + 2(\cos\pi\cos t - \sin\pi\sin t) = 2 - 2\cos t \\ y - 3 = 2\sin(\pi + t) & \Rightarrow \quad y = 3 + 2(\sin\pi\cos t + \cos\pi\sin t) = 3 - 2\sin t \end{cases}$$

Example. Parametrise an ellipse centred at $(0, 0)$, crossing the x axis at ± 2 and y axis at $\pm\dfrac{1}{3}$.



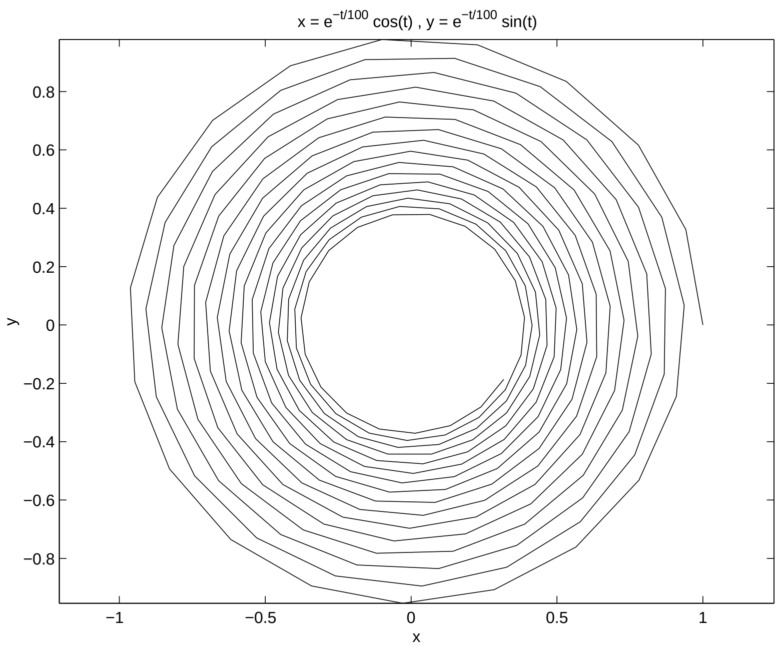
The equation of ellipse is $\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{\frac{1}{3}}\right)^2 = 1$.

Now let $\dfrac{x}{2} = \cos t$ and $3y = \sin t$. Then

$$x = 2\cos t \quad \text{and} \quad y = \frac{1}{3}\sin t.$$

Often curves that cannot be represented as $y = f(x)$ are represented parametrically. (Even the circle $y = \pm\sqrt{1-x^2}$ is strictly speaking not a function as y is not unique.) The classic example is the *spiral*, which can be thought of as a circle with a variable radius; eg $x(t) = r(t)\cos t$ and $y(t) = r(t)\sin t$.

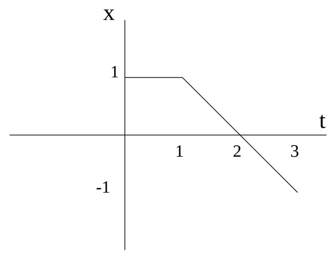
If $r(t) = e^{-\frac{t}{100}}$ the radius tends to 0 as $t \to +\infty$.



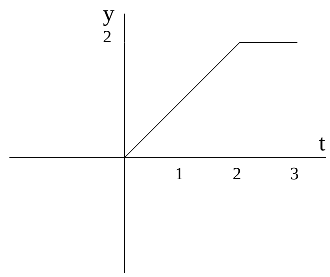
A *helix* is a circle in (x, y) space, which moves up (and down) in z. Say $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = t$. Note none of these parametrisations are unique, $x = \cos \omega t$, $y = \sin \omega t$, $z = \omega t$ parametrises the same helix.

Parametrisations of straight line paths

Suppose $x(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 2 - t & 1 \leq t \leq 3 \end{cases}$

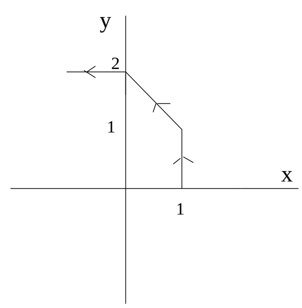


and $y(t) = \begin{cases} t & 0 \leq t < 2 \\ 2 & 2 \leq t \leq 3 \end{cases}$



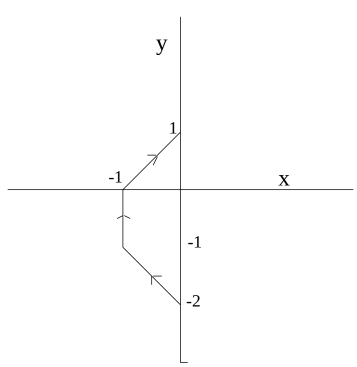
Then

for	$0 \leq t < 1$	$x(t) = 1$	and	$y(t) = t \Rightarrow x = 1$
for	$1 \leq t \leq 2$	$x(t) = 2 - t$	and	$y(t) = t \Rightarrow y = 2 - x$
for	$2 \leq t \leq 3$	$x(t) = 2 - t$	and	$y(t) = 2 \Rightarrow y = 2$



If $x(t)$ and $y(t)$ are composed of straight line segments then you can find a parametrisation so that $x(t)$ and $y(t)$ are too.

Parametrise the following curve:



For $t = 0$ we have $x(0) = 0$ and $y(0) = -2$.

For $0 \le t < 1$ we have $y = -x - 2$. So let $x = -t$ then $y = t - 2$.

For $1 \le t < 2$ we have $x = -1$ and $y = -1 + (t - 1) = -2 + t$.

For $2 \le t \le 3$ we have $y = x + 1$. So let $x = -1 + (t - 2) = t - 3$ then $y = t - 2$. Therefore,

$$x = \begin{cases} -t & \text{for } \; 0 \le t < 1 \\ -1 & \text{for } \; 1 \le t < 2 \\ t - 3 & \text{for } \; 2 \le t \le 3 \end{cases} \qquad \text{and} \qquad y = t - 2 \quad \text{for } \; 0 \le t \le 3.$$

Lines can be parametrised systematically using vectors. For example $y = 1 - x$ passes through (0, 1) and is parallel to the vector $\hat{\imath} - \hat{\jmath}$.

To parametrise the line let $\hat{r}_0 = 0\hat{\imath} + \hat{\jmath}$ be the position vector of the point (0, 1). Then any point on the line $y = 1 - x$ is given by a vector

$$\hat{r} = \hat{r}_0 + t(\hat{\imath} - \hat{\jmath}) \quad \text{for some } t \quad \Rightarrow \quad \hat{r} = \hat{\jmath} + t(\hat{\imath} - \hat{\jmath}) = t\hat{\imath} + (1 - t)\hat{\jmath}.$$

In general if $\hat{r}_0$ is the position vector of a point on the line and $\hat{v}$ is a vector parallel to the line any point on the line is given by

$$\hat{r} = \hat{r}_0 + t\hat{v} \quad \text{for some } t.$$

Example. Find the intersection point of the line, passing through (0, 1, 0) and parallel to the vector $\hat{v} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$, with the hyperboloid

$$x^2 + \left(\frac{y}{2}\right)^2 - x^2 = 1.$$

First parametrise the line using vectors $r_0 = \hat{\jmath}$ and $\hat{v}$. Any point on the line is given by

$$\hat{r} = \hat{r}_0 + t\hat{v} \quad \Rightarrow \quad \hat{r} = \hat{\jmath} + t(2\hat{\imath} - \hat{\jmath} + \hat{k}) = 2t\hat{\imath} + (1 - t)\hat{\jmath} + t\hat{k}.$$

So $x = 2t$, $y = 1 - t$ and $z = t$. Now the intersection points with the hyperboloid can be found as follows:

$$(2t)^2 + \left(\frac{1 - t}{2}\right)^2 - t^2 = 1 \Rightarrow 13t^2 - 2t - 3 = 0 \Rightarrow t = \frac{1 \pm 2\sqrt{10}}{13} \quad \text{two intersection points.}$$