5. Parametrisation of Curves and Line Integrals

5.2 Position vectors, velocity and acceleration

Suppose you are in the car again, where your position in time is $(x(t),\, y(t))$. So your position vector is

$$r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} = a\cos\omega t\hat{\imath} + a\sin\omega t\hat{\jmath}.$$

The position vector of any point on the line passing through (0, 1, 3) which is parallel to the vector $2\hat{\imath} - 3\hat{\jmath} + \frac{\hat{k}}{2}$ is $\hat{r}(t) = \hat{r}_0 + t\hat{v}$, where $\hat{r}_0 = 0\hat{\imath} + \hat{\jmath} + 3\hat{k}$ and $\hat{v} = 2\hat{\imath} - 3\hat{\jmath} + \frac{\hat{k}}{2}$. So

$$\hat{r}(t) = 2t\hat{\imath} + (1 - 3t)\hat{\jmath} + (3 + \frac{t}{2})\hat{k}.$$

If $(x(t),\, y(t),\, z(t))$ is a parametrisation of a path, its position vector is $\hat{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$.

Velocity vectors

The velocity vector of a moving particle $\hat{v}$ must have magnitude $\|\hat{v}\|$ which equals the speed of the particle and direction which equals the direction of motion. So the velocity vector is tangent to the path of motion.

If $\hat{r}(t)$ is the position vector on the path, then

$$\Delta\hat{r} = \hat{r}(t + \Delta t) - \hat{r}(t)$$

is approximately tangent to the curve traced out by $\hat{r}(t)$. The approximation gets better as $\Delta t \to 0$.

Suppose we define

$$\hat{v} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta r}}{\Delta t}$$

then $\hat{v}$ is tangent to the path of motion.

In component term $\hat{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$$\Rightarrow \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k}$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

which has magnitude $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} =$ speed!

So we define the *velocity vector*

$$\hat{v}(t) = \frac{d\hat{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}.$$

Similarly, the *acceleration vector* can be defined as follows:

$$\hat{a}(t) = \frac{d\hat{v}}{dt}.$$

If an object moves with constant velocity that means constant speed and direction ie. in a straight line.

Say $\hat{v} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and at $t = 0$ object passes through $(-2, 1, 0)$ so that $\hat{r}_0 = -2\hat{\imath} + \hat{\jmath}$.

Then the position at any later time t is

$$r(t) = \hat{r}_0 + t\hat{v} = (-2 + 2t)\hat{\imath} + (1 + 3t)\hat{\jmath} + 4t\hat{k}.$$

Example. Find the position and velocity vector of a car moving on a helical path;

$$x = \cos 5t\,, \quad y = \sin 5t\,, \quad z = \frac{t}{2}.$$

$$\hat{r}(t) = \cos 5t\hat{\imath} + \sin 5t\hat{\jmath} + \frac{t}{2}\hat{k}$$

$$\hat{v}(t) = \frac{d\hat{r}}{dt} = -5\sin 5t\hat{\imath} + 5\cos 5t\hat{\jmath} + \frac{1}{2}\hat{k}.$$

Example. Imagine you spin a stone on a string above your head so that it moves in a circle, radius 1m, and 2m above the ground at a constant speed with period π seconds. If the string breaks find the position, velocity and acceleration vectors of the stone after the string breaks.

Before the string breaks

$$\begin{aligned} \hat{r}(t) &= \cos 2t\hat{\imath} + \sin 2t\hat{\jmath} + 2k \\ \hat{v}(t) &= -2\sin 2t\hat{\imath} + 2\cos 2t\hat{\jmath}. \end{aligned}$$

So $\hat{r}(0) = \hat{\imath} + 2\hat{k}$ and $\hat{v}(0) = 2\hat{\jmath}$.

After the string breaks the stone moves under gravity

$$m\hat{a}(t) = -mg\hat{k} \quad \Rightarrow \quad \hat{a}(t) = -g\hat{k} \quad \Rightarrow \quad \frac{d\hat{v}}{dt} = -g\hat{k}.$$

So

$$\begin{aligned}
\hat{v} &= -gt\hat{k} + \hat{v}(0) = 2\hat{\jmath} - gt\hat{k} \\
\frac{d\hat{r}}{dt}(t) &= \hat{v} \quad \Rightarrow \quad \hat{r}(t) = 2t\hat{\jmath} - \frac{1}{2}gt^2\hat{k} + \hat{r}(0) \\
\hat{r}(t) &= \hat{\imath} + 2t\hat{\jmath} + (2 - \frac{1}{2}gt^2)\hat{k}.
\end{aligned}$$

Quiz. Which position vector fits which curve?

(1) $\hat{r}(t) = -t\hat{\imath} + (\frac{3}{2} - t)\hat{\jmath}$

(2) $\hat{r}(t) = \hat{\imath} + (-1 + t)\hat{\jmath}$

(3) $\hat{r}(t) = -\frac{1}{2}\sin t\hat{\imath} + \frac{1}{2}\cos t\hat{\jmath}$

(4) $\hat{r}(t) = 2\cos 3t\hat{\imath} + 2\sin 3t\hat{\jmath}$