5. Parametrisation of Curves and Line Integrals

5.3 Line Integrals and Work Done

Arc length is an example of a line integral

Imagine moving on a curve with position vector $r(t)$. Then the length of the curve is the same as the distance travelled. Suppose the distance travelled after time t is $s(t)$. Then your speed at time t is $\frac{ds}{dt}(t)$.

But if we know $r(t)$ then the speed is $\|\hat{v}(t)\| = \left\|\frac{d\hat{r}}{dt}\right\|$. So

$$\textit{the distance travelled} = \int_{t \text{ at } A}^{t \text{ at } B} \|\hat{v}(t)\| dt = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

which is also the *arc length*.

Example. Find the length of the helix

$$x = \cos 5t\,, \quad y = \sin 5t\,, \quad z = \frac{t}{2}, \quad 0 \le t \le 2\pi.$$

Since $\hat{r}(t) = \cos 5t\hat{\imath} + \sin 5t\hat{\jmath} + \frac{t}{2}\hat{k}$ it follows that

$$\begin{aligned} \hat{v}(t) &= -5\sin 5t\hat{\imath} + 5\cos 5t\hat{\jmath} + \frac{1}{2}\hat{k} \\ \|\hat{v}(t)\| &= \sqrt{25\sin^2 5t + 25\cos^2 5t + \frac{1}{4}} = \sqrt{25\frac{1}{4}} = \frac{\sqrt{101}}{2}. \end{aligned}$$

$$\text{Distance travelled} = \int \|\hat{v}(t)\| dt = \int_0^{2\pi} \frac{\sqrt{101}}{2} dt = \pi\sqrt{101}.$$

Example. Find the length of the spiral

$$\begin{aligned} x(t) &= e^{-\frac{t}{10}}\cos t \quad \text{and} \quad y(t) = e^{-\frac{t}{10}}\sin t \quad \text{for } t \ge 0. \\ \hat{v}(t) &= \left(-\frac{1}{10}e^{-\frac{t}{10}}\cos t - e^{-\frac{t}{10}}\sin t\right)\hat{\imath} + \left(-\frac{1}{10}e^{-\frac{t}{10}}\sin t + e^{-\frac{t}{10}}\cos t\right)\hat{\jmath} \\ \|\hat{v}(t)\| &= \sqrt{\frac{1}{100}e^{-\frac{t}{5}} + e^{-\frac{t}{5}}} = \frac{\sqrt{101}}{10}e^{-\frac{t}{10}} \end{aligned}$$

$$
\begin{aligned}
\text{Arc length} &= \int_0^\infty \|\hat{v}(t)\|\, dt = \frac{\sqrt{101}}{10}\int_0^\infty e^{-\frac{t}{10}} dt \\
\text{Arc length} &= \lim_{A\to\infty} \frac{\sqrt{101}}{10}\int_0^A e^{-\frac{t}{10}} dt \\
&= \lim_{A\to\infty} \frac{\sqrt{101}}{10}\left(-10e^{-\frac{t}{10}}\right]_0^A \\
&= \lim_{A\to\infty} \sqrt{101}\left(1-e^{-\frac{A}{10}}\right) = \sqrt{101}.
\end{aligned}
$$

Example. A ball rolls down a parabolic hill

$$y = x^2 \quad \text{from} \quad x = 2 \text{ to } 0.$$

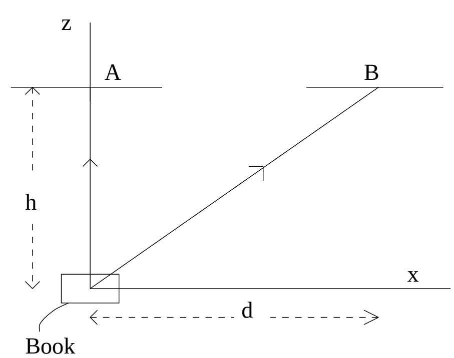
First we parametrise the path. Let $x = t$ then $y = t^2$ for $0 \le t \le 2$. So

$$\hat{r}(t) = t\hat{\imath} + t^2\hat{\jmath} \quad \Rightarrow \quad \hat{v}(t) = \hat{\imath} + 2t\hat{\jmath}$$

$$\Rightarrow \|\hat{v}(t)\| = \sqrt{1+(2t)^2} \quad \Rightarrow \quad \text{Arc Length } = \int_0^2 \sqrt{1+(2t)^2}dt.$$

Work done. To calculate work done you also need to integrate along a path.

How much work does it take to put a heavy book on shelf A or shelf B?



The path to shelf B is longer but work is only done against a force. Work = Force × distance travelled in the direction of the force.

Here the force is gravity $F = -mg\hat{k}$ which acts vertically and both shelf A and B are the same height above the book. To work it out exactly we use vectors.

Let $\Delta\hat{r}_A$ be the vector position of shelf A from the book. Suppose that

$$\Delta\hat{r}_A = h\hat{k}$$

Let r_B be the vector position of shelf B from the book. Suppose that

$$\Delta\hat{r}_B = d\hat{\imath} + h\hat{k}.$$

Recall the *dot product between any two vectors.* Let

$$\hat{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k} \quad \text{and} \quad \hat{w} = w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}.$$

Since $\hat{\imath}.\hat{\jmath} = 0$, $\hat{\imath}.\hat{k} = 0$, $\hat{\jmath}.\hat{k} = 0$ and $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$ as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are mutually perpendicular unit vectors, we obtain

$$\hat{v}.\hat{w} = v_1w_1 + v_2w_2 + v_3w_3.$$

Geometrically, $\hat{v}.\hat{w} = \|v\|\,\|w\|\cos\theta$ where θ is the angle between $\hat{v}$ and $\hat{w}$. So $\hat{v}.\hat{w}$ equals the component of $\hat{v}$ in the $\hat{w}$ direction multiplied by the magnitude of $\hat{w}$. Here we are interested in the component of the distance in the direction of the force.

Work done using vectors

In vector form if you move a book in a straight line a distance $\Delta\hat{r}$ the work done is

$$\hat{F}.\Delta\hat{r}.$$

In the example $\Delta\hat{r}_A = h\hat{k}$ and $\Delta\hat{r}_B = d\hat{\imath} + h\hat{k}$ and $F = -mg\hat{k}$.
So Work done

$$\begin{aligned} w_A &= -mg\hat{k}.h\hat{k} = -mgh \\ w_B &= -mg\hat{k}.(d\hat{\imath} + h\hat{k}) = 0 - mgh = -mgh \quad \text{the same!} \end{aligned}$$

But what if we do not move the book in a straight line? Then we *approximate* the oriented curve (curve with a direction attached) by *line segments.* Divide up the curve at points with position vectors $\hat{r}_0$, $\hat{r}_1$, $\hat{r}_2$, ... , $\hat{r}_n$. The line segments that approximate the curve are

$$\Delta\hat{r}_1 = \hat{r}_1 - \hat{r}_0\,, \quad \Delta\hat{r}_2 = \hat{r}_2 - \hat{r}_1, \ldots\,, \quad \Delta\hat{r}_n = \hat{r}_n - \hat{r}_{n-1}.$$

Then the total work done is a sum over these line segments

$$\begin{aligned} \text{Work done} &\approx \hat{F}(\hat{r}_1).\Delta\hat{r}_1 + \hat{F}(\hat{r}_2).\Delta\hat{r}_2 + \ldots + \hat{F}(\hat{r}_n).\Delta\hat{r}_n \\ &= \sum_{i=1}^{n} \hat{F}(\hat{r}_i).\Delta\hat{r}_i. \end{aligned}$$

But this is only approximate. To get the actual value let $\|\Delta\hat{r}\| \to 0$ and $n \to +\infty$. Then the sum becomes an integral

$$\text{Work done} = \lim_{\substack{\|\Delta\hat{r}_i\|\to 0 \\ n\to\infty}} \sum_{i=1}^{n} \hat{F}(\hat{r}_i).\Delta\hat{r}_i = \int_C \hat{F}(r).d\hat{r}$$

which is a line integral of the vector field $\hat{F}$ along the oriented curve C.

Example. Find the work done by moving a book mass 2kg along an arc of the circle $x^2 + z^2 = h^2$ from $(-h, 0, 0)$ to $(0, 0, h)$.

The force is gravity $\hat{F} = -mg\hat{k} = -2g\hat{k}$ and $\hat{r}(t) = x(t)\hat{\imath} + z(t)\hat{k}$, where $x(t)$ and $z(t)$ parametrise the path.

Here

$$\begin{cases} x(t) = h\cos(\pi - t) = -h\cos t \\ z(t) = h\sin(\pi - t) = h\sin t, \end{cases}$$

where $0 \leq t \leq \frac{\pi}{2}$. So $\hat{r} = -h\cos t\hat{\imath} + h\sin t\hat{k}$.

Now $\hat{F}.d\hat{r} = \hat{F}.\frac{d\hat{r}}{dt}dt$ and $\frac{d\hat{r}}{dt} = h\sin t\hat{\imath} + h\cos t\hat{k}$.

So $\hat{F}.\frac{d\hat{r}}{dt} = -2gh\cos t$.

$$\begin{aligned} \text{Work done} &= \int_0^{\frac{\pi}{2}} \hat{F}.d\hat{r} = \int_0^{\frac{\pi}{2}} \hat{F}.\frac{d\hat{r}}{dt}dt \\ &= \int_0^{\frac{\pi}{2}} -2gh\cos t dt = (-2gh\sin t]_0^{\frac{\pi}{2}} = -2gh. \end{aligned}$$

Conservative Fields

If $\int_A^B \hat{F}.d\hat{r}$ is *independent* of the path taken, then $\hat{F}$ is called a *conservative field.*

Gravity is a conservative field because

$$\int_A^B \hat{F}.d\hat{r} = -mg\int_A^B dz = -mg(z(B) - z(A))$$

and it does not matter how you get from A to B.

Any constant field must be conservative by the same reasoning.