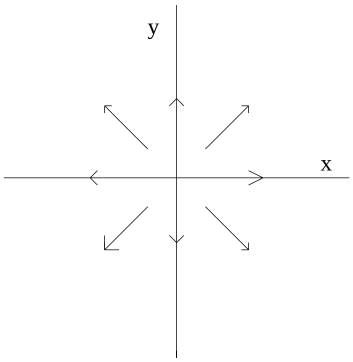
5. Parametrisation of Curves and Line Integrals

5.4 Gradient Fields

Take the field $\hat{F} = x\hat{\imath} + y\hat{\jmath}$, which you can picture as a set of vectors $\hat{F}(x,\, y)$ at the point $(x,\, y)$.

$$\begin{aligned} \hat{F}.d\hat{r} &= (x\hat{\imath} + y\hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath}) \\ &= xdx + ydy \end{aligned}$$



So

$$\begin{aligned} \int_A^B \hat{F}.d\hat{r} &= \int_A^B xdx + ydy \\ &= \int_A^B d\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = \left(\frac{x^2}{2} + \frac{y^2}{2}\right)\Big/_{B} - \frac{(x^2+y^2)}{2}\Big/_{A}. \end{aligned}$$

Once again $\hat{F}$ is a conservative field. But it was chosen in rather a special way. $\hat{F}$ is a *gradient field* ie. $\hat{F} = \nabla f$ for some f.

$$\hat{F} = \nabla\left(\frac{x^2}{2} + \frac{y^2}{2}\right)\ ,\ \text{recal } \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}.$$

Any gradient field is a conservative field and so is path independent.
To show this let $\hat{F} = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$ then $\hat{F}.d\hat{r} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$ (recall chain rules). So

$$\int_A^B \hat{F}.d\hat{r} = \int_A^B df = f(B) - f(A).$$

Finding the work done against a conservative field really comes down to finding an f such that $\hat{F} = \nabla f$.

Example. Find the work done against the Electric Field

$$\hat{E} = \frac{\alpha(x\hat{\imath} + h\hat{\jmath} + z\hat{k})}{x^2 + y^2 + z^2}$$

along any path from (1, 0, 0) to (1, 1, 1).

E is a gradient field because

$$E = \nabla f \quad \text{where} \quad f = \frac{\alpha}{2}\ln|x^2 + y^2 + z^2|$$

since

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \\ &= \frac{\alpha x\hat{\imath} + \alpha y\hat{\jmath} + \alpha z\hat{k}}{(x^2 + y^2 + z^2)} = E.\end{aligned}$$

So the path will not matter.

$$\begin{aligned}\int E.d\hat{r} = \int \nabla f.d\hat{r} &= \int \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \int d\left(\frac{\alpha}{2}\ln|x^2 + y^2 + z^2|\right).\end{aligned}$$

So

$$\begin{aligned}\text{Work done } = \left(\frac{\alpha}{2}\ln|x^2 + y^2 + z^2|\right]_{(1,0,0)}^{(1,1,1)} &= \frac{\alpha}{2}(\ln 3 - \ln 1) \\ &= \frac{\alpha \ln 3}{2}.\end{aligned}$$

Example. Evaluate work done against the force

$$F = \nabla f \quad \text{where} \quad f = xy \quad \text{from } (0,\, 0) \text{ to } (1,\, 2).$$

$$\begin{aligned}\text{Work done } = \int \hat{F}.d\hat{r} &= \int \nabla f.d\hat{r} \\ &= \int df = (xy]_{(0,0)}^{(1,2)} = 2 - 0 = 2.\end{aligned}$$

But sometimes you may have to find f.

Example. Show that $F = (y + x)\hat{\imath} + x\hat{\jmath}$ is a gradient field.
If so $F = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$ for some f.

$$\Rightarrow \frac{\partial f}{\partial x} = y + x \quad \text{and} \quad \frac{\partial f}{\partial y} = x.$$

So

$$\int \frac{\partial f}{\partial x}\, dx = \int (y + x)\, dx \quad \Rightarrow \quad f = yx + \frac{x^2}{2} + g(y).$$

But then $\dfrac{\partial f}{\partial y} = x + \dfrac{dg}{dy}$ which must equal x. So $\dfrac{dg}{dy} = 0$ and g is any constant (f is not unique). Take $g = 0$ so that $f = yx + \dfrac{x^2}{2}$ then $\hat{F} = \nabla f$ and it is a gradient field.

Example. Show that $\hat{F} = \dfrac{(x+y)}{2}\hat{\imath} + \dfrac{y}{2}\hat{\jmath}$ is NOT a gradient field.
If $\hat{F} = \nabla f$ then

$$\frac{\partial f}{\partial x} = \frac{x+y}{2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{y}{2}$$
$$\Rightarrow \quad f = \frac{x^2}{4} + \frac{xy}{2} + g(y) \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{2} + \frac{dg}{dy}.$$

But this can never equal $\dfrac{y}{2}$. So $\hat{F}$ is not a gradient field.