5. Parametrisation of Curves and Line Integrals

5.5 Work done or path integrals for nonconservative fields

Path integrals for Forces which are not conservative depend on the path. To work them out we need to parametrise the path by t say, and convert the integral to one over t. Say $\hat{F}(x, y) = F_1(x, y)\hat{\imath} + F_2(x, y)\hat{\jmath}$ is a non conservative field and $x(t)$ and $y(t)$ provide a parametrisation of the path. Then

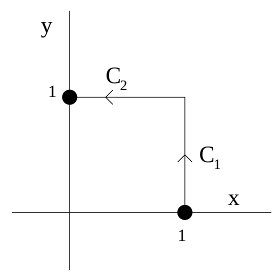
$$\hat{F}.d\hat{r} = F_1(x, y)dx + F_2(x, y)dy$$

$$\int_C \hat{F}.d\hat{r} = \int_{t=a}^{t=b} \left(F_1(x(t), y(t))\frac{dx}{dt} + F_2(x(t), y(t))\frac{dy}{dt} \right) dt.$$

Example. Evaluate $\int_C \hat{F}.d\hat{r}$ where $\hat{F} = \left(\frac{x+y}{2}\right)\hat{\imath} + \frac{y}{2}\hat{\jmath}$ and the path is as shown.
Split up the path into two parts:

along C_1 : $x = 1, \quad y = t, \quad 0 \le t \le 1$

along C_2 : $x = 2 - t, \quad y = 1, \quad 1 \le t \le 2.$



Then

$$\int_C \hat{F}.d\hat{r} = \int_{C_1} \hat{F}.d\hat{r} + \int_{C_2} \hat{F}.d\hat{r}.$$

Now along C_1 : $r = \hat{\imath} + t\hat{\jmath}$ so $d\hat{r} = \hat{\jmath}\,dt$. And along C_2 : $r = (2-t)\hat{\imath} + \hat{\jmath}$ so $d\hat{r} = -\hat{\imath}\,dt$.
So

$$\int_{C_1} \hat{F}.d\hat{r} = \int_{t=0}^{1} \left(\frac{(1+t)}{2}\hat{\imath} + \frac{t}{2}\hat{\jmath} \right) .\hat{\jmath}\,dt = \int_{t=0}^{1} \frac{t}{2}dt = \left.\frac{t^2}{4}\right]_0^1 = \frac{1}{4}$$

and

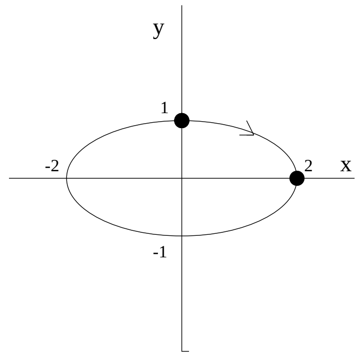
$$\begin{aligned} \int_{C_2} \hat{F}.d\hat{r} &= \int_{t=1}^{2} \left(\frac{(3-t)}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} \right) .(-\hat{\imath}\,dt) = \int_1^2 \frac{(-3+t)}{2}dt \\ &= \left(\frac{-3t}{2} + \frac{t^2}{4} \right]_1^2 = -2 + \frac{3}{2} - \frac{1}{4} = -\frac{1}{2} - \frac{1}{4}. \end{aligned}$$

Therefore, $\int_C \hat{F}.d\hat{r} = -\frac{1}{2}$.

Example. Evaluate $\int_C \hat{F}.d\hat{r}$ were $F = (x+y)\hat{\imath} + y\hat{\jmath}$ and C is the arc of the ellipse $\frac{x^2}{4} + y^2 = 1$ from $(0, 1)$ to $(2, 0)$.

First parametrise the path:

$$\frac{x}{2} = \cos t\,, \quad y = \sin t \quad \text{for} \quad t = \frac{\pi}{2} \text{ to } 0.$$



Then along C

$$\hat{F} = (2\cos t + \sin t)\hat{\imath} + \sin t\hat{\jmath}$$

and

$$d\hat{r} = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}\right) dt = (-2\sin t\hat{\imath} + \cos\hat{\jmath})dt.$$

So

$$\begin{aligned}
\int_C \hat{F} d\hat{r} &= \int_{\frac{\pi}{2}}^{0} \left((2\cos t + \sin t)\hat{\imath} + \sin t\hat{\jmath}\right).(-2\sin t\hat{\imath} + \cos t\hat{\jmath})dt \\
&= \int_{\frac{\pi}{2}}^{0} \left(-4\sin t\cos t - 2\sin^2 t + \cos t\sin t\right) dt \\
&= \int_{\frac{\pi}{2}}^{0} \left(-3\sin t\cos t - 2\sin^2 t\right) dt.
\end{aligned}$$

To evaluate we need to use double angle formulae

$$\sin 2a = 2\sin a\cos a \quad \text{and} \quad \cos 2a = \cos 2a - \sin 2a = 1 - 2\sin^2 a.$$

$$\begin{aligned}
\int_C \hat{F}.d\hat{r} &= \int_{\frac{\pi}{2}}^{0} \left(\frac{-3}{2}\sin 2t + \cos 2t - 1\right) dt \\
&= \left(\frac{3}{4}\cos 2t + \frac{1}{2}\sin 2t - t\right]_{\frac{\pi}{2}}^{0} \\
&= \frac{3}{4} - \left(\frac{-3}{4} - \frac{\pi}{2}\right) = \frac{3}{2} + \frac{\pi}{2}.
\end{aligned}$$