On Ramsey Numbers for Sets Free of Prescribed Differences

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Abstract

For a positive integer d, a set S of positive integers is *difference d-free* if $|x-y| \neq d$ for all $x, y \in S$. We consider the following Ramsey-theoretical question: Given $d, k, r \in \mathbf{Z}^+$, what is the smallest integer n such that every r-coloring of $[1, n]$ contains a monochromatic k-element difference d-free set? We provide a formula for this n. We then consider the more general problem where the monochromatic k-element set must avoid a given *set* of differences rather than just one difference.

Keywords: Difference-free sets, integer Ramsey theory, monochromatic sets

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1 Introduction

For d a positive integer, a set S of integers is called *difference d-free* if for all $x, y \in S$, $|x - y| \neq d$. Interesting results concerning the number of difference d-free subsets of $[1, n] = \{1, 2, ..., n\}$ and some generalizations and variants are given in [2-6,9]. In this work, we consider, for a given d, the Ramsey-theoretical question (posed in [7]) of how large n must be to guarantee that under any partition of $[1, n]$ into r subsets, some subset must contain a k-element difference d-free set. Ramsey problems with a somewhat similar flavor may be found in [1] and [8].

Denote by $F_d(k; r)$ the smallest integer n such that every r-coloring of $[1, n]$ contains a monochromatic k-element difference d-free set. In the next section, we will prove that

$$F_d(k;r) = r(k-1) + d\left\lfloor \frac{r(k-1)}{d} \right\rfloor + 1.$$

In Section 3 we consider a generalization of the function F_d; namely, rather than looking for monochromatic sets that are difference d-free for a single d, we are concerned with monochromatic sets that are free of all differences belonging to a given set D. In Section 4, we present some open questions and conjectures.

2 Solution to $F_d(k;r)$

We begin this section with some terminology.

An r-coloring χ of a set S of positive integers is called *(d,k)-valid* (or simply *valid* if d and k are understood) if it does not contain a monochromatic k-element difference d-free set. A (d,k)-valid coloring of an interval $[1,n]$ is called *maximal* if there is no (d,k)-valid coloring of $[1,n+1]$. Given a positive integer d, a pair of integers x and y such that $|x-y| = d$ is called a *d-pair*.

Given a coloring χ of a set S of positive integers and a positive integer d, we may build a partition of S as follows. Let m_0 be the least member of S. If $m_0+d \in S$ and $\chi(m_0) = \chi(m_0+d)$, let $S_0 = \{m_0, m_0+d\}$; otherwise let $S_0 = \{m_0\}$. If $i \geq 1$ and S_{i-1} has been defined and $S \neq \bigcup_{j=0}^{i-1} S_j$, let m_i be the least element of $S - \bigcup_{j=0}^{i-1} S_j$. If $m_i + d \in S$ and $\chi(m_i) = \chi(m_i+d)$, let $S_i = \{m_i, m_i+d\}$; otherwise let $S_i = \{m_i\}$. Repeat this until all members of S have been assigned to some S_i. It is clear that the sets S_i do, in fact, form a partition of S. We shall denote by $p_d(\chi, S)$ the number of i such that $|S_i| = 2$, and by $q_d(\chi, S)$ the number of i such that $|S_i| = 1$. If χ is monochromatic on a set S, we will denote these numbers simply as $p_d(S)$ and $q_d(S)$; that is, $p_d(S)$ is the number of disjoint pairs of elements of S with difference d, and $q_d(S) = |S| - 2p_d(S)$. Note that for any χ and S, $2p_d(\chi, S) + q_d(\chi, S) = |S|$.

Lemma 1 *Let $S \subseteq \mathbf{Z}^+$ and $d \in \mathbf{Z}^+$. The largest difference d-free subset of S has size $p_d(S) + q_d(S)$.*

Proof. Let $p = p_d(S)$ and let $m = p + q_d(s)$. Let $A_1, A_2, \ldots, A_p$ be disjoint d-pairs in S. Let $M = S - \{\max(A_i) : 1 \leq i \leq p\}$. Note that $|M| = m$. For any $x, x+d \in S$, then there is some i for which either $A_i = \{x-d, x\}$ or $A_i = \{x, x+d\}$. Hence, one of x and $x+d$ equals $\max(A_i)$, so x and $x+d$ are not both in M. Thus M is difference d-free.

If T is a subset of S with more than m elements, then by the pigeonhole principle, some A_i contains more than one element of T. But then T contains a d-pair, so T is not difference d-free. Thus, the largest difference d-free set has size m. □

Lemma 2 *Let $d, k, r \in \mathbb{Z}^+$ and let $n = F_d(k; r) - 1$. If χ is a (d,k)-valid r-coloring of $[1, n]$, then*

$$p_d(\chi, [1, n]) + q_d(\chi, [1, n]) = r(k-1).$$

proof Let χ be a (d,k)-valid r-coloring of $[1, n]$. For each color i, let $S_i = \{x \in [1, n] \mid \chi(x) = i\}$, $p_i = p_d(S_i)$, and $q_i = q_d(S_i)$. Let

$$p = \sum_{i=1}^{r} p_i = p_d(\chi, [1, n]) \text{ and } q = \sum_{i=1}^{r} q_i = q_d(\chi, [1, n]).$$

Since χ is a maximal valid coloring, the largest difference d-free subset of each S_i has size $k-1$, and therefore by Lemma 1, $p_i + q_i = k-1$ for each i. Summing this equation over i, $1 \leq i \leq r$, we have that $p + q = r(k-1)$ as desired. □

Lemma 3 *Let $d, k, r \in \mathbb{Z}^+$, let $n = F_d(k; r) - 1$. If χ is a (d,k)-valid r-coloring of $[1, n]$, then*

$$p_d(\chi, [1, n]) = d \left\lfloor \frac{r(k-1)}{d} \right\rfloor.$$

Proof. Let χ be a (d,k)-valid coloring of $[1, n]$, and let $p = p_d(\chi, [1, n])$, $q = q_d(\chi, [1, n])$.

First, suppose $d \leq n < 2d$. Let $c_0 = \chi(n-d+1)$. If we extend χ to $[1, n+1]$ by assigning $n+1$ the color c_0, then $[1, n+1]$ does not contain a monochromatic k-element difference d-free set, for if A were such a set, then clearly it would be in color c_0, and its largest element would be $n+1$. But then $n+1-d \notin A$, and hence $(A - \{n+1\}) \cup \{n+1-d\}$ would also be difference d-free (since $n+1-2d \leq 0$) contradicting the meaning of $F_d(k; r)$. Hence, it is not possible that $d \leq n < 2d$.

If $n < d$, then no d-pairs can exist in $[1, n]$, so $p = 0$, and $r(k-1) = q = n < d$ by Lemma 2, so $d\left\lfloor \frac{r(k-1)}{d} \right\rfloor = 0$ and we have the desired result.

Thus, we assume that $n \geq 2d$. We will construct another maximal valid coloring χ' from χ. Note that $n = 2p_d(\chi', [1, n]) + q_d(\chi', [1, n])$, and therefore $p = p_d(\chi', [1, n])$ by Lemma 2.

Let m be the largest integer such that $(2d)m+2d \leq n$. Let $I_x = [(2d)x+1, (2d)x+d]$ and $J_x = [(2d)x+d+1, (2d)x+2d]$ for all $x \in [0, m]$. Notice that there is at least one such pair of intervals since $n \geq 2d$. Define χ' as follows. Let $\chi'(i) = \chi(i)$ for each $i \in I_x$. Now define $\chi'(j) = \chi(j-d)$ for each $j \in J_x$. Then $\{i, i+d\}$ is a monochromatic d-pair under χ' for each $i \in I_x$, and therefore, since there are no d-pairs in I_x, we have $p_d(\chi', I_x \cup J_x) = d$ and $q_d(\chi', I_x \cup J_x) = 0$.

If $n > 2d(m+1)$, let $H = [(2d)m+2d+1, \min((2d)m+3d, n)]$; otherwise, let $H = \varnothing$. Define $\chi'(h) = \chi(h)$ for all $h \in H$. We claim that χ' is (d,k)-valid on $[1, \max(H)]$. To see this, let $I = (\bigcup_{x=0}^{m} I_x) \cup H$ and let $J = \bigcup_{x=0}^{m} J_x$. Assume A is a difference d-free set that is monochromatic under χ'. Since $\chi(i) = \chi'(i)$ for all $i \in I$, χ' is valid on I, and hence $|A \cap I| < k$. By the way χ' is defined on J, and the fact that A contains at most one member of $\{i, i+d\}$ for each i, we have $|A| < k$. This proves the claim.

Note that $n < 2dm + 3d$, so that χ' is valid on $[1, n]$. If this were not the case, then by coloring each

$$j \in [(2d)m + 3d + 1, (2d)m + 4d] \text{ by } \chi'(j) = \chi'(j - d)$$

we would be extending χ' to a valid coloring of $[1, 2dm + 4d]$, which is not possible by the meanings of m and n.

Notice by the way χ' is defined that $p = p_d(\chi', [1, n]) = d(m+1)$. This implies that $q = |H|$ and hence, by the previous paragraph, $q < d$. Hence, since d divides p, by Lemma 2 we have

$$p = d\left\lfloor \frac{p}{d} \right\rfloor = d\left\lfloor \frac{p+q}{d} \right\rfloor = d\left\lfloor \frac{r(k-1)}{d} \right\rfloor.$$

□

Theorem 4 *For all positive integers d, k, and r,*

$$F_d(k; r) = r(k-1) + d\left\lfloor \frac{r(k-1)}{d} \right\rfloor + 1.$$

Proof. Let $n = F_d(k;r) - 1$, let χ be a valid coloring of $[1,n]$, and let $p = p_d(\chi,[1,n])$, $q = q_d(\chi,[1,n])$. Then

$$n = 2p + q = (p+q) + p = r(k-1) + d\left\lfloor \frac{r(k-1)}{d} \right\rfloor$$

by Lemmas 2 and 3, proving the result. □

3 Avoiding a Set of Differences

Having solved the problem for a single d, we now consider the more general problem of finding, for a given set of positive integers D, the Ramsey numbers for sets which avoid all differences $d \in D$.

Given a set D of positive integers, denote by $F_D(k;r)$ the smallest integer n such that every r-coloring of $[1,n]$ contains a monochromatic k-element set that is difference d-free for each $d \in D$. We will say that this k-element set is *difference D-free*. For $r = 1$, we denote the function by $F_D(k)$; that is, $F_D(k)$ is the least n such that $[1,n]$ contains a k-element difference D-free set.

The next theorem provides an upper bound for $F_D(k;r)$ for a large class of sets D, in particular for all finite D.

Proposition 5 *Let D be a set of positive integers. If there is a least positive integer m such that $m \nmid d$ for all $d \in D$, then for all positive integers k and r,*

$$F_D(k;r) \leq mr(k-1) + 1.$$

Proof. We need to show that any r-coloring of $[1, mr(k-1)+1]$ contains a k-element difference D-free set. By the pigeonhole principle, an r-coloring of $[1, mr(k-1)+1]$ must have some color c with more than $m(k-1)$ elements. Likewise, among the elements with color c, there is some congruence class modulo m to which at least k integers belong. Since no pair of these k integers have difference in D, there is a monochromatic k-element set that is difference D-free. □

Corollary 6 *For all $k, r \in \mathbb{Z}^+$, if $[1,n] \subseteq D \subseteq \mathbb{Z}^+$ and $(n+1) \nmid d$ for all $d \in D$, then $F_D(k;r) = (n+1)r(k-1) + 1$.*

Proof. Let $t = (n+1)r(k-1)$. By Proposition 5, $F_D(k;r) \leq t+1$.

For each i, $1 \leq i \leq r$, let $S_i = [(i-1)(n+1)(k-1)+1, i(n+1)(k-1)]$, so that $\bigcup_{i=1}^{r} S_i = [1,t]$. Define the r-coloring χ on $[1,t]$ by $\chi(S_i) = i$ for each i. Let A be any monochromatic difference D-free subset of S. Hence, for some j, $A \subseteq S_j$, and therefore, since each consecutive pair of elements of A must differ by at least $n+1$,

$$|A| \leq 1 + \lfloor \frac{(n+1)(k-1)-1}{n+1} \rfloor = k-1.$$

Thus χ is (d,k)-valid on $[1,t]$ for all $d \in D$, which proves that $F_D(k;r) \geq t+1$. □

As a special case of Corollary 6, we have the following.

Corollary 7 *For all n, k, $r \in \mathbb{Z}^+$, $F_{[1,n]}(k;r) = (n+1)r(k-1)+1$.*

Unlike the situation with the classical Ramsey-type numbers, such as van der Waerden or Schur numbers, the threshold function for difference D-free sets is not trivial in the setting of only one color. In fact, $F_D(k;r) \leq F_D(r(k-1)+1)$ for all D, k, and r, which follows immediately from the following simple proposition since, for a given D, the family of all D-free sets is hereditary (i.e, every subset of a difference D-free set is difference D-free).

Proposition 8 *Let $\mathcal{S}$ be a hereditary family of sets. Let $R(\mathcal{S},k;r)$ be the least positive integer n such that every r-coloring of $[1,n]$ has a monochromatic k-element member of $\mathcal{S}$. Then $R(\mathcal{S},k;r) \leq R(\mathcal{S},r(k-1)+1;1)$.*

Proof. Let $m = R(\mathcal{S},r(k-1)+1;1)$. So $[1,m]$ contains an $(r(k-1)+1)$-element set S where $S \in \mathcal{S}$. Suppose, for a contradiction, that $R(\mathcal{S},k;r) > m$. So there is a valid r-coloring χ of $[1,m]$, i.e., no color contains a k-element member of $\mathcal{S}$. For each i, $1 \leq i \leq r$, let $S_i = \{x \in S : \chi(x) = i\}$. Since $\mathcal{S}$ is hereditary and $S \in \mathcal{S}$, each $S_i \in \mathcal{S}$. Since χ is valid, $|S_i| \leq k-1$ for each i. Then

$$|S| = \sum_{i=1}^{r} |S_i| \leq r(k-1),$$

a contradiction. □

We are able to give exact values for $F_D(k)$ for certain choices of D.

Proposition 9 *Let $k, n \in \mathbb{Z}^+$ and let $D = \{1, n\}$. Then*

$$F_D(k) = \begin{cases} 2(k-1)+1 & \text{if } n \text{ is odd} \\ 2(k-1) + \left\lfloor \frac{2(k-1)}{n} \right\rfloor + 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is odd, $F_D(k) \geq F_1(k) = 2(k-1)+1$ by Theorem 4, and $F_D(k) \leq 2(k-1)+1$ by Proposition 5.

Now let n be even. Let $b = \left\lfloor \frac{2(k-1)}{n} \right\rfloor$. To show that $F_D(k) \geq 2(k-1)+b+1$, we show that in $[1, 2(k-1)+b]$ there is no k-element difference D-free set. If $b = 0$, this is obvious, so assume $b \geq 1$.

Let $X = \{x_1 < x_2 < \cdots < x_k\} \subseteq [1, 2(k-1)+b]$, and let $d_i = x_{i+1} - x_i$ for $1 \leq i \leq k-1$. We may assume $d_i \neq 1$ for all i. Since $\sum_{i=1}^{k-1} d_i \leq 2(k-1)+b-1$, there are at most $b-1$ of the d_i's that are greater than 2. Hence, by the pigeonhole principle, somewhere within the sequence $\{d_i\}_{i=1}^{k-1}$, there must exist at least

$$\left\lceil \frac{k-1-(b-1)}{b} \right\rceil$$

consecutive 2's. Now,

$$\left\lceil \frac{k-1-(b-1)}{b} \right\rceil = \left\lceil \frac{k}{b} - 1 \right\rceil \geq \frac{n}{2}\left(\frac{k}{k-1}\right) - 1 > \frac{n}{2} - 1.$$

Therefore there exist at least $n/2$ consecutive d_i's that equal 2. Thus, X is not difference n-free.

Finally, we show that $F_D(k) \leq 2(k-1)+b+1$ for n even. Let

$$S = \left\{2i + \left\lfloor \frac{2i}{n} \right\rfloor + 1 : 0 \leq i \leq k-1\right\}.$$

It is easy to check that S is a k-element difference D-free set. Hence, since $S \subseteq [1, 2(k-1)+b+1]$, we have $F_D(k) \leq 2(k-1)+b+1$. □

Proposition 10 *Let $b > a \geq 1$, let $k \in \mathbb{Z}^+$, and let $D = [a, b]$. Then $F_D(k) = k + b\left\lfloor \frac{k-1}{a} \right\rfloor$.*

Proof. Let $t = \left\lfloor \frac{k-1}{a} \right\rfloor$. If $t = 0$, it is easy to see that $F_D(k) = k$, so we may assume $t \geq 1$.

Let $A_i = [(a+b)i+1, (a+b)i+a]$ for each i, $0 \le i \le t-1$, and let $A_t = [(a+b)t+1, k+bt]$. Let $A = \bigcup_{i=0}^{t} A_i$. Then $|A| = ta + |A_t| = k$. Clearly, A is difference D-free, and hence $F_D(k) \le k+bt$.

We claim that for each i, $1 \le i \le t$, there is no difference D-free subset of $S_i = [(a+b)(i-1)+1, (a+b)i]$ with more than a elements. If this is true, then since t is the largest integer such that $(a+b)t \le k-1+bt$, the size of any difference D-free set in $[1, k-1+bt]$ is at most $(at)+(k-1+bt)-(a+b)t = k-1$, thereby proving $F_D(k) \ge k+bt$.

To prove the claim, let $X = \{x_1 < x_2 < \cdots < x_\ell\}$ be a difference D-free subset of S_i. We may assume $x_1 = (a+b)(i-1)+1$, since otherwise a translation would produce another ℓ-element difference D-free set whose first element is $(a+b)(i-1)+1$. Let $Y = [x_1+a, x_1+b]$. Clearly, no member of Y belongs to X. Also,

$$S_i - Y = \{x_1\} \cup \left(\bigcup_{j=x_1+1}^{x_1+a-1} \{j, j+b\} \right).$$

Since for each $j \in [x_1, x_1+a-1]$, at most one member of $\{j, j+b\}$ can belong to X, at most a members of $S_i - Y$ belong to X. □

4 Remaining Questions

Based on computer calculations, we believe the following conjecture is true, which would generalize Theorem 4, Corollary 7, and Proposition 10.

Conjecture 1 *If $D = [a, b]$, then*

$$F_D(k; r) = r(k-1) + b \left\lfloor \frac{r(k-1)}{a} \right\rfloor + 1.$$

We also suspect that the inequality of Proposition 8 is actually an equality when $\mathcal{S}$ is the family of D-free sets, i.e., that the following holds.

Conjecture 2 *For any set of positive integers D and for each k, $r \in \mathbb{Z}^+$,*

$$F_D(k; r) = F_D(r(k-1)+1).$$

When $|D| = 1$ or $D = [1, n]$ for some positive integer n, then Conjecture 2 holds by Theorem 4 and Corollary 7, respectively.

Let us say that two sets D and E are F-*equivalent* if $F_D(k; r) = F_E(k; r)$ for all $k, r \in \mathbb{Z}^+$. The following result provides one example of an F-equivalence.

Proposition 11 *Let $a \in \mathbb{Z}^+$, let S be a set of odd positive integers with $1 \in S$, and let $D = \{as \mid s \in S\}$. Then for all $k, r \in \mathbb{Z}^+$*

$$F_D(k; r) = F_a(k; r).$$

Proof. Since $a \in D$, it suffices to show

$$F_D(k; r) \leq F_a(k; r). \tag{1}$$

for all k and r. We first show (1) for $r = 1$. Consider the following k-element difference a-free set in $[1, F_a(k)]$:

$$I = [1, a] \cup [2a+1, 3a] \cup \ldots \cup \left[\left(2\left\lfloor\frac{k-1}{a}\right\rfloor - 2\right)a + 1, \left(2\left\lfloor\frac{k-1}{a}\right\rfloor - 1\right)a\right]$$
$$\cup \left[\left(2\left\lfloor\frac{k-1}{a}\right\rfloor\right)a + 1, \left(2\left\lfloor\frac{k-1}{a}\right\rfloor\right)a + \left(k - a\left\lfloor\frac{k-1}{a}\right\rfloor\right)\right].$$

(Note: this is the set I from the proof of Lemma 3.)

It is easy to see that for any pair of elements of I with difference ma, m must be even. Since D contains only odd multiples of a, I is difference D-free. So (1) holds when $r = 1$.

Since the proposition holds for $r = 1$, by using Theorem 4 we have

$$F_D(r(k-1)+1) = F_a(r(k-1)+1) = F_a(k; r) \tag{2}$$

for any r. By Proposition 8, $F_D(k; r) \leq F_D(r(k-1)+1)$ which, together with (2), gives (1). □

We would like to know what the minimal F-equivalent subsets of a given set D are; in other words, which subsets S of D have the property that S is F-equivalent to D but no proper subset of S is F-equivalent to D? Particularly, can we describe the minimal F-equivalent subsets of $[a, b]$? Moreover, we would like to determine all non-F-equivalent subsets of $[1, n]$, i.e., to find the equivalence classes of $2^{[1,n]}$ under this equivalence relation.

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