Avoiding Monochromatic Sequences with Gaps in a Fixed Translation of the Primes

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Abstract

For a set D of positive integers, a sequence $\{a_1 < a_2 < \cdots < a_k\}$ is called an *k-term D-diffsequence* if $a_i - a_{i-1} \in D$ for all $i \in \{2, \ldots, k\}$. For a positive integer r, a set of positive integers D is *r-accessible* if every r-coloring of $\mathbb{Z}^+$ has arbitrarily long monochromatic D-diffsequences. The largest r such that D is r-accessible is called the *degree of accessibility* of D. It is already known that each odd translation of the set of primes, $P+t$, is 2-accessible. We offer new results on the accessibility of translations the primes. The main result is that for any $c \geq 2$, the degree of accessibility of $P+c$ does not exceed the smallest prime factor of c.

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1 Introduction

Many results in Ramsey theory take the following form, where $\mathcal{F}$ is some specific family of sets of integers and $r \in \mathbb{Z}^+$: for every r-coloring of $\mathbb{Z}^+$ there are arbitrarily long monochromatic members of $\mathcal{F}$. One classical theorem of this type is van der Waerden's theorem which takes $\mathcal{F}$ to be the family of arithmetic progressions, and which holds for all r [4]. Brown, Graham, and Landman [1] considered a strengthening of van der Waerden's theorem by restricting the family of arithmetic progressions $\{a, a+d, \ldots, a+kd\}$ to those for which d must belong to some prescribed set of positive integers D. They referred to this property of D as "largeness."

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A question posed by Tom Brown was whether there exists some translation of the primes, $P + c = \{p + c : p \text{ is prime}\}$ such that $P + c$ is 2-large (i.e., for every 2-coloring of $\mathbb{Z}^+$ there are arbitrarily long monochromatic arithmetic progressions with gaps in $P + c$).

In a recent paper [3], the authors again considered Ramsey properties for sequences with consecutive differences belonging to a prescribed set D; however, they removed the requirement that the sequences be arithmetic progressions. For a set D of positive integers, a sequence $a_1 < a_2 < \cdots a_k$ is a called a *k-term D-diffsequence* if $a_i - a_{i-1} \in D$ for all $i \in \{2, \ldots, k\}$. For a positive integer r a set of positive integers D is *r-accessible* if every r-coloring of $\mathbb{Z}^+$ has arbitrarily long monochromatic D-diffsequences. The largest r such that D is r-accessible is called the *degree of accessibility* of D, denoted by doa(D). It is obvious that r-accessibility is a weaker condition than r-largeness. Jungic [2] has shown that there exist sets that are r-accessible for all r but that are not r-large for all r, thereby answering another question of Brown.

The primary focus in [3] was on the situation where D is a translation of the primes, motivated by Brown's aforementioned corresponding question on largeness. There it was shown that for c odd and positive, doa$(P + c) \geq 2$. It was also shown that P is not 3-accessible. The paper did not give any results on the degree of accessibility of even translations of P. In this article we give an upper bound on doa$(P + c)$ for all $c \geq 2$. We also make some related observations.

2 Upper Bound on the Degree of Accessibility

We are able to give an upper bound on the degree of accessibility of $P + c$ for each $c \geq 2$. In particular, no even translation of P is 3-accessible.

Theorem 2.1 *Let $c \geq 2$ and let q be the smallest prime factor of c. Then doa$(P + c) \leq q$.*

Proof. We consider three cases. In each case we give an explicit $(q+1)$-coloring of $\mathbb{Z}^+$ that does not have arbitrarily long monochromatic $(P + c)$-diffsequences.

Case 1: $c = q^n t$ where $t \geq 3$ and $(q, t) = 1$.
Let $Y = \{y \in \mathbb{Z}^+ : y \equiv jt (\text{mod } tc) \text{ for some } j,\ 0 \leq j \leq (q - 1)\}$. Define $\chi : \mathbb{Z}^+ \to \{0, 1, 2, \ldots, q\}$ as follows:

$$\chi(i) = \begin{cases} b & \text{if } i \equiv b (\text{mod } q),\ 0 \leq b \leq q - 1, \text{ and } i \notin Y \\ q & \text{if } i \in Y \end{cases}$$

To complete the proof of this case, we show that if a $(P+c)$-diffsequence has more than $k = q^{n-1}t^2$ terms, then it is not monochromatic under χ. For a contradiction, assume $S = \{s_1, s_2, \ldots, s_{k+1}\}$ is a monochromatic $(P + c)$-diffsequence.

First assume $\chi(S) = b$ for some $b \leq q - 1$. Then any two consecutive terms of S, say x and $x+p+c$, differ by a multiple of q, which implies that $p = q$. Hence S is an arithmetic progression with gap $q + c$. Now, $(q + c, tc) = q$, so that for every multiple aq of q, some element of S is congruent to $b+aq$ modulo tc. Also, since $(q,t)=1$, some member of $\{0,t,2t,\ldots,(q-1)t\}$ is congruent to b modulo q. Hence, some member of S has color q, a contradiction.

Now assume $\chi(S)=q$. Note that no two consecutive terms in S are congruent modulo tc, since c composite implies there is no prime p for which $p+c\equiv 0(\text{mod } ct)$. Since there are only q different congruence classes modulo tc to which members of S may belong, for some $j\in\{1,2,\ldots,q\}$ there exist u and v such that $s_j\equiv ut \pmod{tc}$, $s_{j+1}\equiv vt \pmod{tc}$, and $0\le v<u\le q-1$. Thus there is a prime p such that

$$p+c\equiv (v-u)t \pmod{tc}.$$

Since $t>1$, this implies $t=p$. Hence $pc|[p+c+(u-v)p]$, which is impossible because $p(1+u-v)+c\le pq+c\le 2c<pc$.

Case 2: $c=q^n$, $c\neq 2$.
Let $\alpha=\Pi_{i=0}^{q-1}(c+i(q^{n-1}+1))$. Let

$$X=\{x\in\mathbb{Z}^+ : x\equiv j(q^{n-1}+1) \pmod{\alpha} \text{ for some } j, 0\le j\le q-1\}$$

.

Define the $(q+1)$-coloring χ as follows:

$$\chi(i)=\begin{cases} b & \text{if } i\equiv b(\text{mod } q),\ 0\le b\le q-1, \text{ and } i\notin X\\ q & \text{if } i\in X\end{cases}$$

Let S be a $(P+c)$-diffsequence with length at least α/q.

First, assume S has color b, where $0\le b\le q-1$. Then any two consecutive terms of S must differ by a multiple of q. Hence, S must be an arithmetic progression with gap $q+c=q+q^n$.

We next show that

$$(q+q^n,\alpha)=q. \tag{1}$$

We have

$$(q^n+q,\alpha)=q\left(q^{n-1}+1, q^{n-1}\Pi_{j=1}^{q-1}\left(q^n+j(q^{n-1}+1)\right)\right).$$

Hence, to prove (1) it suffices to have, for each j, $1\le j\le q-1$,

$$\left(q^{n-1}+1, q^n+j(q^{n-1}+1)\right)=1. \tag{2}$$

Note that (2) easily follows from the fact that

$$qn+j(q^{n-1}1+1)=(q+j)(q^{n-1}+1)-q$$

and our assumption that if $n=1$ then $q\neq 2$.

Because of (2) and the size of S, we see that for every positive integer k, some element of S is congruent to $b+kq$ modulo α. Also, since some element of $\{j(q^{n-1}+1) : 0 \leq j \leq q-1\}$ is congruent to $b (\text{mod } q)$, this implies that some element of S has color q, which is a contradiction.

Now assume that S has color q. Note that no two consecutive terms of S are congruent modulo α because otherwise, for some prime p, we have

$$p + c \equiv 0 (\text{mod} \alpha) \tag{3}$$

By the definition of α, (3) would imply that $c|p$, i.e., that $c = p = q$. But then $q(q+2)|\alpha$ and (by (3)) $\alpha|2q$, a contradiction.

Since no two consecutive terms of S are congruent modulo α, and since the members of S belong to at most q different congruence classes modulo α, there exist two consecutive members of S, say x and y, such that $x \equiv u(q^{n-1}+1) \ (\text{mod } \alpha)$ and $y \equiv v(q^{n-1}+1) (\text{mod } \alpha)$ where $0 \leq v < u \leq q-1$. Hence there is a prime p such that

$$\alpha | (p+z),$$

where $z = c + (u-v)(q^{n-1}+1))$. Since $z|\alpha$, this implies $z|p$ and hence $z = p$. Therefore, since $c|\alpha$, we have $c|2p$. By assumption, $c > 2$, so $c = p$ or $c = 2p$, both of which contradict the fact that $z = p$.

Case 3: $c = 2$.
Let $A = \{x \in \mathbb{Z}^+ : x \equiv 0, 1, 7, 8, \text{ or } 14 (\text{mod } 24)\}$. Define the coloring χ on $\mathbb{Z}^+$ as follows:

$$\chi(i) = \begin{cases} 0 & \text{if } i \in A \\ 1 & \text{if } i \notin A \text{ and } i \text{ even} \\ 2 & \text{if } i \notin A \text{ and } i \text{ odd} \end{cases}$$

In colors 1 and 2, all gaps between monochromatic pairs are even, so that the only gap belonging to $P+2$ is 4. It follows easily that the longest $(P+2)$-diffsequence in color 1 is five, and likewise for color 2.

If $\chi(i) = \chi(j) = 0$ and $j - i \in P+2$, then the only possible ordered pairs of residue classes modulo 24 that i and j, in that order, can belong to, are (0,1), (0,7), (1,8), (1,14), (7,8), and (7,14). From this it follows that no $(P+2)$-diffsequence in color 0 has length exceeding three. □

3 Some Questions and Remarks

We mention the following open questions and observations:

1. Is the upper bound of Theorem 1 the true value of $\text{doa}(P+c)$ for $c > 1$?

2. We believe P is 2-accessible although we do not have a proof. We are sure that there is no periodic 2-coloring that can be used to disprove this claim. To see this, let α be a 2-coloring of $\mathbb{Z}^+$ with period m. We may assume there is an $i \in \{1, \ldots, m-1\}$ such that $\alpha(i) = \alpha(i+1)$, since otherwise the set of all even numbers would form a monochromatic infinite P-diffsequence. Also, by Dirichlet's theorem, there are an infinite number of primes congruent to 1 modulo m and an infinite number of primes congruent to -1 modulo m. Hence there exists an infinite sequence $x_1 < y_1 < x_2 < y_2 < \cdots$ such that each x_j is congruent to i modulo m, each y_j is congruent to $i+1$ modulo m, and where all gaps $y_j - x_j$ and $x_j - y_{j-1}$ are prime. Hence there is an infinite monochromatic P-diffsequence.

3. Is $P+1$ is r-accessible for all r? If not, what is $\mathrm{doa}(P+1)$?

4. For a set D that is not r-accessible, denote by $m(D, r)$ the largest integer k such that every r-coloring of $\mathbb{Z}^+$ admits a monochromatic k-term D-diffsequence. By the proof of Case 3 of Theorem 1, we see that $m(P+2, 3) \leq 5$. In [3] it was shown that $m(P, 3) \leq 8$. An exhaustive computer search shows that $m(P, 3) \geq 5$. On the other hand, the following 3-coloring of $\mathbb{Z}^+$ avoids monochromatic 6-term P-diffsequences.

$$\chi(i) = \begin{cases} 0 & \text{if } i \equiv 0, 3, 6, \text{ or } 9 \pmod{18} \\ 1 & \text{if } i \equiv 1, 5, 7, 11, 13, 15, \text{ or } 17 \pmod{18} \\ 2 & \text{if } i \equiv 2, 4, 8, 10, 12, 14, \text{ or } 16 \pmod{18} \end{cases}$$

Thus we know that $m(P, 3) = 5$. We are able to find exact values of $m(P+c, 3)$ for other small even integers c using this approach, but would like to be able to to extend this to $m(P+c, q+1)$ for general $c \geq 2$, where q is defined as in Theorem 1.

We conclude with a table summarizing what is known about the degrees of accessibility of the translations of P. Here q represents the smallest prime dividing c.

c	$\mathrm{doa}(P+c)$
0	1 - 2
1	≥ 2
>1, even	1 - 2
>1, odd	2 - q

Table 1 Degree of Accessibility of $P+c$

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