The chromatic number of $K^2(9, 4)$ is 11

Abdollah Khodkar and David Leach
Department of Mathematics
University of West Georgia
Carrollton, GA 30118

Abstract

In 2004, Kim and Nakprasit showed that the chromatic number of $K^2(9, 4)$ is at least 11. In this note we present an 11-coloring for $K^2(9, 4)$. This proves that the chromatic number of $K^2(9, 4)$ is 11.

Keywords: Kneser graph, square, graph coloring

1 Introduction

For a simple graph $G$, let $G^2$ be the square of $G$, obtained from $G$ as follows: the vertex set $V(G^2)$ is $V(G)$, and two distinct vertices $u, v \in V(G^2)$ are adjacent if and only if the distance between $u$ and $v$ in $G$ is at most 2. We use [4] for terminology and notation which are not defined here.

Let $[n] = \{1, 2, \ldots, n\}$. We denote the family of $k$-element subsets of $[n]$ by $\binom{[n]}{k}$. For $n \geq 2k$, the vertex set of the Kneser graph $K(n, k)$ is $\binom{[n]}{k}$, and two vertices $A$ and $B$ are adjacent in $K(n, k)$ if and only if $A \cap B = \emptyset$. When $n = 2k + 1$, the graph $K(2k + 1, k)$ is also called an odd graph. Two vertices $A$ and $B$ of $K^2(2k + 1, k)$ are adjacent if and only if $A \cap B = \emptyset$ or $|A \cap B| = k - 1$. The chromatic number of a graph $G$, written $\chi(G)$, is the minimum number of colors needed to color the vertices so that adjacent vertices receive different colors. An independent set in a graph is a set of pairwise nonadjacent vertices. The independent number of a graph $G$, written $\alpha(G)$, is the maximum size of an independent set of vertices. In [1] it was proved that $\alpha(K^2(9, 4)) = 12$. Note that two vertices $A$ and $B$ of $K^2(9, 4)$ are nonadjacent if and only if $|A \cap B| = 1$ or 2. Using the fact that $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$, we have $\chi(K^2(9, 4)) \geq 11$, as it was noted in [1].

2 The main result

Since $K(5, 2)$ is the Petersen graph, $K^2(5, 2)$ is a complete graph on 10 vertices. Hence, $\chi(K^2(5, 2)) = 10$. In [1] it was shown that $\chi(K^2(7, 3)) = 6$. Figure 1 displays
a partition of $V(K^2(9, 4))$ into eleven independent sets ($\mathcal{I}_j, j \in [11]$). This proves that:

**Theorem 1** $\chi(K^2(9, 4)) = 11$.

Using nauty [3], we see that $\mathcal{I}_i \cong \mathcal{I}_j$ for $i, j \in [7]$. In particular,

- $\mathcal{I}_2 = (1, 3, 2)(4, 9, 6)(7, 5, 8)\mathcal{I}_1$
- $\mathcal{I}_3 = (5, 8, 6)(1, 7, 9, 4, 2)\mathcal{I}_1$
- $\mathcal{I}_4 = (3, 8, 4, 7, 5, 9)\mathcal{I}_1$
- $\mathcal{I}_5 = (6, 7)(1, 9, 8, 3, 4, 2)\mathcal{I}_1$
- $\mathcal{I}_6 = (5, 8)(1, 4, 3, 9, 7, 6, 2)\mathcal{I}_1$
- $\mathcal{I}_7 = (1, 2)(4, 9, 7, 8, 5, 6)\mathcal{I}_1$

![Figure 1: A partition of $\binom{[9]}{4}$ into eleven independent sets.](image)

**Sketch of our search.**

1. Start with the following seven disjoint Steiner triple systems of order 9 (see [2]).
\[ \mathcal{S}_1 = \{124, 139, 158, 167, 236, 257, 289, 345, 378, 468, 479, 569\} \]
\[ \mathcal{S}_2 = \{129, 134, 156, 178, 235, 248, 267, 368, 379, 457, 469, 589\} \]
\[ \mathcal{S}_3 = \{127, 135, 146, 189, 238, 249, 256, 347, 369, 458, 479, 569\} \]
\[ \mathcal{S}_4 = \{126, 138, 147, 159, 239, 245, 278, 346, 357, 489, 568, 679\} \]
\[ \mathcal{S}_5 = \{125, 137, 148, 169, 234, 268, 279, 356, 389, 459, 467, 578\} \]
\[ \mathcal{S}_6 = \{123, 149, 157, 168, 247, 258, 269, 348, 359, 367, 456, 789\} \]
\[ \mathcal{S}_7 = \{128, 136, 145, 179, 237, 246, 259, 349, 358, 478, 567, 689\} \]

2. For each \( i \in [7] \) and \( j \in [12] \), add \( x_{ij} \) to block \( B_j \in \mathcal{S}_i \) in such a way that the resulting set of twelve 4-subsets of \([9]\) is an independent set in \( K^2(9, 4) \). Computer searches show that there are precisely 288 independent sets in \( K^2(9, 4) \) containing \( \mathcal{S}_i \) for each \( i \in [7] \). All \( 2016(=7 \times 288) \) of these are isomorphic.

3. Find seven disjoint independent sets among these 2016 independent sets. Let \( R \) be the remaining 42 4-subsets of \([9]\) which do not appear in these seven independent sets.

4. Use computer searches to partition \( R \) into four independent sets in \( K^2(9, 4) \). If such a partition exists, then we have 11 independent sets in \( K^2(9, 4) \) which partition \( V(K^2(9, 4)) \). Otherwise, go to Step (3) and generate a new set of seven disjoint independent sets.

References


