Lemma 1. Let $\triangle ABC$ be a triangle and let $D$ and $E$ be the midpoints of sides $AC$ and $BC$. Then segment $DE$ is parallel to segment $AB$. Conversely, let $D$ be the midpoint of segment $AC$ and $E$ any point on segment $BC$. If $DE$ is parallel to segment $AB$, then $E$ is the midpoint of segment $BC$. (See Figure 1.)

Figure 1: Figure for Lemma 1

Proof. Let $\triangle ABC$ be a triangle and let points $D$ and $E$ be the midpoints of segments $AC$ and $BC$. Extend $DE$ past $E$ to a point $F$ so that $DE$ is congruent to $EF$ and construct segment $BF$. See Figure 2. Since $D$ and $E$ are the midpoints of segments $AC$ and $BC$, respectively, we have $AD$ is congruent to $CD$ and $BE$ is congruent to $EC$. Since the vertical angles $\angle CED$ and $\angle BEF$ are congruent, we have that triangles $\triangle CED$ and $\triangle BEF$ are congruent by SAS. Hence, $CD$ and $BF$ are congruent. By the transitivity of congruence, $AD$ and $BF$ are congruent.
Secondly, since $\triangle CED$ and $\triangle BEF$ are congruent, $\angle CDE$ is congruent to $\angle BFE$. Since these two angles are alternate interior angles with respect to the transversal $DF$, it follows that segments $AD$ and $BF$ are parallel.

**Figure 2: Construction for Lemma 1**

Construct segment $BD$. Since $BD$ is a transversal to the parallel segments $AD$ and $BF$, $\angle FBD$ and $\angle ADB$ are congruent.

Since $AD$ and $BF$ are congruent, $BD$ is congruent to itself, and $\angle FBD$ and $\angle ADB$ are congruent, triangles $\triangle ABD$ and $\triangle FDB$ are congruent by SAS. Hence, $\angle ABD$ and $\angle FDB$ are congruent. Since these are alternate interior angles with respect to the transversal $BD$ and the segments $DE$ and $AB$, we see that these two segments must be parallel.

Conversely, suppose $D$ is the midpoint of segment $AC$, $E$ is a point on $BC$, and $DE$ is parallel to segment $AB$.

Since $\angle CDE$ and $\angle CAB$ are corresponding angles with respect to the transversal $AC$, they are congruent. Angle $\angle C$ is congruent to itself, so triangles $\triangle ABC$ and $\triangle DEC$ are similar. It follows that

$$\frac{CE}{BC} = \frac{CD}{AC} = \frac{1}{2}.$$ 

It follows that $E$ is the midpoint of segment $BC$.

**Lemma 2.** Let $\triangle PQR$ be a right triangle with right angle at $Q$. Let $W$ be the midpoint of the hypotenuse $PR$. Then segments $PW$, $RW$, and $QW$ are all congruent.

**Proof.** Let $\triangle PQR$ be a right triangle with right angle at vertex $Q$. Let $W$ be the midpoint of the hypotenuse $PR$ and construct the segment $QW$. Since $W$ is the midpoint of $PR$, we have $PW$ and $RW$ are congruent.

Construct the altitude from $W$ to segment $QR$ and let $V$ be the foot of this altitude. Since $W$ is the midpoint of segment $PR$ and the segment $VW$ is parallel to segment $PQ$, $V$ must be the midpoint of $QR$, by Lemma 1. Thus, $QV$ is congruent to segment $VR$.  

2
Since $QV$ is congruent to segment $VR$, $WR$ is congruent to itself, and angles $\angle QVW$ and $\angle RVW$ are right angles and therefore congruent, we have triangle $\triangle QVW$ is congruent to triangle $\triangle RVW$, by SAS. Hence, $QW$ and $RW$ are congruent.

Hence, segments $PW$, $RW$, and $QW$ are all congruent.

**Theorem 3.** Let $\triangle ABC$ be a triangle. Let $L$, $M$, and $N$ be the midpoints of the sides $BC$, $AC$, and $AB$, respectively. Let $D$, $E$, and $F$ be the feet of the altitudes from the vertices $A$, $B$, and $C$, respectively, to the opposite sides of $\triangle ABC$. It is known that altitudes of the triangle are concurrent, meeting in the orthocenter $H$ of the triangle. Let $X$, $Y$, and $Z$ be the midpoints of the segments $AH$, $BH$, and $CH$, respectively.

Then the points $L$, $M$, $N$, $D$, $E$, $F$, $X$, $Y$, $Z$ lie on a circle, the nine-point circle of the triangle $\triangle ABC$.

**Proof.** Let $\triangle ABC$ be a triangle. Let $L$, $M$, and $N$ be the midpoints of the side $BC$, $AC$, and $AB$, respectively.

Next, construct the segments $LM$, $LN$ and $MN$, and their perpendicular bisectors. We remark that these perpendicular bisectors are concurrent for the following reason. Any point on the perpendicular bisector of the segment $LM$ is equally distant from the points $L$ and $M$. Similarly, any point on the perpendicular bisector of the segment $LN$ is equally distant from the points $L$ and $N$. The two bisectors certainly meet in a point $U$. Since $U$ is on both these perpendicular bisectors, $U$ is equally distant from all three points, $L$, $M$, and $N$. Since $U$ is equally distant from points $M$ and $N$, it lies on the perpendicular bisector of the segment $MN$. Hence, the three perpendicular bisectors meet at the point $U$, which is equally distant from the vertices of $\triangle LMN$. That is, $U$ is the center of the
circumcircle of $\triangle LMN$.

Construct the altitudes of $\triangle ABC$ from the vertices $A$, $B$, and $C$, to the respective sides of $\triangle ABC$, meeting the sides at points $D$, $E$, and $F$, respectively. It is known that these altitudes are concurrent, the intersection being the orthocenter $H$ of triangle $\triangle ABC$. Let $X$, $Y$, and $Z$, be the midpoints of the segments $AH$, $BH$, and $CH$, respectively. See Figure 4.

The claim is that $L$, $M$, $N$, $D$, $E$, $F$, $X$, $Y$, and $Z$ lie on a circle, which is the circumcircle of triangle $\triangle LMN$ with center $U$.

Note that since $M$ is the midpoint of segment $AC$ and $X$ is the midpoint of segment $AH$, the segment $MX$ connects the midpoints of two sides of the triangle $\triangle ACH$, and is therefore parallel to the remaining side, segment $CH$, by Lemma 1. Likewise, since $L$ is the midpoint of segment $BC$ and $Y$ is the midpoint of segment $BH$, the segment $LY$ connects the midpoints of two sides of the triangle $\triangle BCH$, and is therefore parallel to the remaining side, segment $CH$, likewise by Lemma 1. By transitivity, segments $MX$ and $LY$ are parallel. See Figure 5.

Note that since $M$ is the midpoint of segment $AC$ and $L$ is the midpoint of segment $BC$, the segment $ML$ connects the midpoints of two sides of the triangle $\triangle ABC$, and is therefore parallel to the remaining side, segment $AB$ by Lemma 1. Likewise, since $X$ is the midpoint of segment $AH$ and $Y$ is the midpoint of segment $BH$, the segment $XY$ connects the midpoints of two sides of the triangle $\triangle ABH$, and is therefore parallel to the remaining side, segment $AB$, likewise by Lemma 1. By transitivity, segments $ML$ and $XY$ are parallel. See Figure 5.

Since $MX$ and $LY$ are parallel to $CH$, these two segments are parallel to $CF$, which
contains \(CH\). But segments \(ML\) and \(XY\) are parallel to \(AB\). Since \(CF\) is the altitude from \(C\) to side \(AB\), \(CF\) is perpendicular to segment \(AB\). It follows that the pair of segments \(MX\) and \(LY\) are perpendicular to segments \(ML\) and \(XY\). That is, quadrilateral \(MLYX\) is a rectangle. See Figure 5.

Since quadrilateral \(MLYX\) is a rectangle, the segments \(MY\) and \(LX\) bisect each other at a point \(U\) into four congruent segments. That is, \(UM\), \(UL\), \(UY\), and \(UX\) are congruent. Note that \(U\) is therefore the midpoint of segment \(MY\).

Note that since \(M\) is the midpoint of segment \(AC\) and \(Z\) is the midpoint of segment \(CH\), the segment \(MZ\) connects the midpoints of two sides of the triangle \(\triangle ACH\), and is therefore parallel to the remaining side, segment \(AH\). Likewise, since \(N\) is the midpoint of segment \(AB\) and \(Y\) is the midpoint of segment \(BH\), the segment \(NY\) connects the midpoints of two sides of the triangle \(\triangle ABH\), and is therefore parallel to the remaining side, segment \(AH\). By transitivity, segments \(MZ\) and \(NY\) are parallel. See Figure 6.

Since \(MZ\) and \(NY\) are parallel to \(AH\), these two segments are parallel to \(AD\), which contains \(AH\). But segments \(MN\) and \(YZ\) are parallel to \(BC\). Since \(AD\) is the altitude from \(A\) to side \(BC\), \(AD\) is perpendicular to segment \(BC\). It follows that the pair of segments \(MZ\) and \(NY\) are perpendicular to segments \(MN\) and \(YZ\). That is, quadrilateral \(MNYZ\) is a rectangle. See Figure 6.
Since quadrilateral $MNYZ$ is a rectangle, the segments $MY$ and $NZ$ bisect each other at the same point $U$ (since $U$ is the unique midpoint of $MY$) into four congruent segments. That is, $UM$, $UN$, $UY$, and $UZ$ are congruent.

Hence, $L$, $M$, $N$, $X$, $Y$, and $Z$ lie on the circle with center $U$.

Consider the right triangle $\triangle XDL$. The point $U$ is the midpoint of the hypotenuse $XL$, so $U$ is equally distant from the three vertices, by Lemma 2. Hence, $UL$ is congruent to $UD$. See Figure 7.
Consider the right triangle $\triangle YEM$. The point $U$ is the midpoint of the hypotenuse $YM$, so $U$ is equally distant from the three vertices, by Lemma 2. Hence, $UM$ is congruent to $UE$. See Figure 8.

![Figure 8: Triangle YEM](image)

Consider the right triangle $\triangle ZFN$. The point $U$ is the midpoint of the hypotenuse $ZN$, so $U$ is equally distant from the three vertices, by Lemma 2. Hence, $UN$ is congruent to $UF$. See Figure 9.

![Figure 9: Triangle ZFN](image)

It now follows that $D, E, F, L, M, N, X, Y,$ and $Z$ lie on the circle with center $U$, as desired. See Figure 10.

$\square$
Figure 10: The Nine Point Circle