THE MISÈRE MONOID OF ONE-HANDED ALTERNATING GAMES

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Abstract

Misère games are notoriously more difficult to analyze than their normal-play counterparts. To deal with the inherent lack of structure, Plambeck (2005) considered restricted universes of games and developed the concept of indistinguishability, modulo a given universe; the resulting quotient semi-group is called the misère monoid. Results for only three general universes are known. We introduce a fourth: the universe of alternating games and their sums. We find that the misère monoid of one-handed alternating games is isomorphic to \( (\mathbb{Z}, +) \).

1. Introduction

Consider the variation Penny Nim of a single-heap game of Nim. Two players Left and Right play with a stack of pennies, which are either all heads-up or all tails-up; Left can play on a tails-up stack and Right can play on a heads-up stack, by removing at least one penny and then inverting any remaining coins. The player who removes the last penny loses. This is an alternating game, since neither Left nor Right can make two consecutive moves in one stack, and it is being played under

\footnote{The first two authors would like to thank NSERC for partial support of this research.}
misère rules. In the present paper, we analyze individual alternating components (e.g., one stack of pennies) in order to answer the following question: under misère play, who wins a set of alternating games when they are played in a disjunctive sum? Note that a disjunctive sum of alternating components is not necessarily itself an alternating game.

For a given game $G$, we use $G^L$ to denote a general left option and $G^L$ to denote the set of all such options. We may refer to a single left option from the position $G^R$ as $G^{RL}$ and to the set of all left options from the position $G^R$ as $G^{RL}$. A game $H$ is a follower of $G$ if $H$ can be reached from $G$ by some sequence of (not necessarily alternating) moves.

**Definition 1.** A game $G$ is said to be alternating if $G^{LL} = \emptyset$ and $G^{RR} = \emptyset$ for all left and right options $G^L, G^R$ of $G$, and if every follower of $G$ is also alternating.

In PENNY NIM, each component is one-handed; that is, only one player has an option from the initial position. We can specify which of Left or Right has the first move by the terms left-handed or right-handed, respectively. Games can also be two-handed alternating, where both players have at least one option from the start ($G^L \neq \emptyset, G^R \neq \emptyset$). If we placed a heads-up or tails-up stack of pennies on its side and allowed either player to move by taking some pennies and orienting the stack appropriately, then PENNY NIM would become a two-handed alternating game. This paper considers one-handed positions; however, the majority of our one-handed results are true in the ‘universe’ of all alternating games. Since every two-handed game has exclusively one-handed followers, an understanding of one-handed games will help us subsequently analyze all alternating games. Readers familiar with normal-play games will notice that one-handed alternating games are restricted to the normal-play values $-1, 0, \text{and } 1$.

It is interesting to note that one-handed or ‘end’ games in general can be very problematic for misère play — Aaron Siegel calls them “significant pathologies”[9] — and yet there is simple, elegant structure in one-handed alternating games.

### 1.1. Background

We begin with an overview of prerequisite material. Conventionally, the players Left and Right are female and male, respectively. We have already encountered an instance of misère-play, where the first player unable to move wins; the more standard ending condition is normal-play, where the first player unable to move loses. Games or positions are defined in terms of their options: $G = \{G^L \mid G^R\}$. The simplest game is the zero game, $0 = \{, \mid ,\}$, where the dot indicates an empty set of options.

In both play conventions, the outcome classes next ($\langle N \rangle$), previous ($\langle P \rangle$), left ($\langle L \rangle$), and right ($\langle R \rangle$) are partially ordered as shown in Figure 1, with Left preferring moves toward the top and Right preferring moves toward the bottom. In our analysis of
alternating games, we often simultaneously consider the misère outcome and normal outcome of a game; to distinguish between the two, we introduce the superscripts \(-\) and \(\+)
, respectively. Thus a game in \(\mathcal{N}^{-}\cap \mathcal{R}^{+}\) is a next-win under misère play and a right-win under normal play. When convenient we also use the outcome functions \(o^{-}(G)\) and \(o^{+}(G)\) to identify the misère or normal outcome of a game \(G\).

\[
\begin{array}{c}
\mathcal{L} \\
\mathcal{P} \\
\mathcal{N} \\
\mathcal{R}
\end{array}
\]

Figure 1: The partial order of outcome classes.

Many definitions from normal-play game theory\(^2\) are used without modification for misère games, including disjunctive sum, equality, and inequality. Thus, for misère games,

\[
G = H \text{ if } o^{-}(G + X) = o^{-}(H + X) \text{ for all games } X,
\]

\[
G \geq H \text{ if } o^{-}(G + X) \geq o^{-}(H + X) \text{ for all games } X.
\]

In normal-play, the negative of a game is defined recursively as \(-G = \{ -G^R | -G^L \}\), and is so-called because \(G + (-G) = 0\) for all games \(G\). Under misère play, however, this property holds only if \(G\) is the zero game \([5]\). To avoid confusion and inappropriate cancellation, we write \(\overline{G}\) instead of \(-G\) and refer to this game as the conjugate of \(G\).

In normal-play games, there is an easy test for equality: \(G = 0\) if and only if \(G \in \mathcal{P}^{+}\), and so \(G = H\) if and only if \(G - H \in \mathcal{P}^{+}\). In misère-play, no such test exists. Equality of misère games is difficult to prove and, moreover, is relatively rare; for example, besides \(\{ \cdot | \cdot \}\) itself, there are no games equal to the zero game under misère play \([5]\). Within the last ten years (see \([7], [8],\) and \([9]\) for example), a partial solution to these challenges has been presented: redefine ‘equality’ by restricting the game universe. This method has been used with much success by \([2], [3],\) and \([4]\). Given a set of games \(\mathcal{X}\), misère equivalence (modulo \(\mathcal{X}\)) is defined by

\[
G \equiv H \pmod{\mathcal{X}} \text{ if } o^{-}(G + X) = o^{-}(H + X) \text{ for all games } X \in \mathcal{X},
\]

while misère inequality (modulo \(\mathcal{X}\)) is defined by

\[
G \geq H \pmod{\mathcal{X}} \text{ if } o^{-}(G + X) \geq o^{-}(H + X) \text{ for all games } X \in \mathcal{X}.
\]

We use the words equivalent and indistinguishable interchangeably, and if \(G \neq H \pmod{\mathcal{X}}\) then \(G\) and \(H\) are said to be distinguishable.

\(^2\)A complete overview of normal-play game theory can be found in \([1]\).
Given a universe $\mathcal{X}$, we can determine the equivalence classes under $\equiv \pmod{\mathcal{X}}$ and form the quotient semi-group $\mathcal{X}/\equiv$. This quotient, together with the tetrapartition of elements into the sets $\mathcal{P}^-, \mathcal{N}^-, \mathcal{R}^-$, and $\mathcal{L}^-$, is called the misère monoid of the universe $\mathcal{X}$, denoted $\mathcal{M}_\mathcal{X}$. The misère monoid is by convention written multiplicatively, and its identity element, denoted $e$, is the equivalence class of zero. As an example, the misère monoid of the games $0, \{0 \mid \cdot\},$ and all possible sums, is given by $\langle e, \alpha \mid \alpha^2 = \alpha \rangle$, with $\mathcal{N} = \{e\}$, $\mathcal{R} = \{\alpha\}$, $\mathcal{P} = \mathcal{L} = \emptyset$ [2]. The closure of a set of games is the set of all disjunctive sums of those games and their followers. We denote the closure of the set of all alternating games by $\mathcal{A}$ and the closure of one-handed alternating games by $\mathcal{O}$. Note that if $\mathcal{X} \subseteq \mathcal{Y}$ then $G \equiv H \pmod{\mathcal{Y}}$ implies $G \equiv H \pmod{\mathcal{X}}$; in particular, equivalence in $\mathcal{A}$ immediately gives equivalence in $\mathcal{O} \subseteq \mathcal{A}$.

Following Ottaway’s PhD thesis [6], we consider the universe $\mathcal{A}$ under misère play and find the misère monoid $\mathcal{M}_\mathcal{O}$ of one-handed alternating games. We would like to acknowledge Meghan Allen for suggesting the harder problem of constructing the monoid for all alternating games, of which this paper is a first step.

In Section 2, we determine the equivalence classes of one-handed alternating games inside the larger universe $\mathcal{A}$, and in Section 3, we calculate outcomes and identify the equivalence classes of $\mathcal{O}$ alone. In Section 4 we present the misère monoid $\mathcal{M}_\mathcal{O}$. Future goals for this research naturally include analyzing two-handed games and determining $\mathcal{M}_\mathcal{A}$. Some progress has been made, but the equivalence classes of two-handed games are more complicated than those for $\mathcal{O}$. It would also be interesting to classify the non-alternating games that can be written as a disjunctive sum of alternating positions; for example, the game $\{* \mid *\}$ can be decomposed into $* + *$. In this way we could extend our knowledge of the alternating universe to a broader family of games. Interested readers should refer to Ottaway’s thesis [6] for an extended discussion of alternating\(^3\) games, including an analysis of subtraction games on coins (a generalization of PENNY NIM).

2. Equivalences

In analyzing alternating games, we find it useful to classify a position by both its misère outcome and normal outcome. In general all $4^2 = 16$ pairs of outcomes can be attained [5] (for example, there exist games which are Left-win under both normal- and misère-play), but more than half of these pairs do no occur among one-handed games. Since either Left or Right has no first move in a one-handed game $G$, that player wins immediately under misère rules and loses immediately under normal rules, and so $o^-\langle G \rangle \neq \mathcal{P}^-, o^+\langle G \rangle \neq \mathcal{N}^+$, and $o^-\langle G \rangle \neq o^+\langle G \rangle$. The remaining seven outcome pairs are each attained in the one-handed universe, as

\(^3\)In [6] and elsewhere, alternating games are referred to as consecutive move ban games.
demonstrated in Figure 2 with the zero game and the games

\[ A = \{0 \mid \cdot\}, B = \{0, \overline{A} \mid \cdot\}, C = \{\overline{B} \mid \cdot\}, \]

along with their corresponding right-handed conjugates\(^4\).

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Figure 2: The possible outcomes pairs for one-handed alternating games.

Most of the positions in Figure 2 appear in PENNY NIM; if we consider stacks on
which Left can play, then a one-penny stack is the game \(A\), a two-penny stack is
the game \(B\), and all other non-zero stacks are equivalent (modulo \(A\)) to \(B\).

Throughout the paper we make repeated use of the fact that a single one-handed
game alternates between being left-handed and right-handed as players alternate
turns. Furthermore, if Left has a good (misère) first move in a left-handed game,
then it is to a position in \(L^-\), because no one-handed game is in \(P^-\). Likewise, if
Right as a good first move in an alternating game, it is to a position in \(R^-\).

We show in Theorem 4 that all one-handed games in \(N^- \cap P^+\) are indistinguishable
from the zero game, an incredibly useful result that actually holds in the larger
universe \(A\) of all alternating games. We then extend this result by establishing that
a single one-handed game is indistinguishable (modulo \(A\)) from any other with the
same outcome pair; in particular, the seven games in Figure 2 are equivalent to all
others in their respective classes.

We first require the following two lemmas.

**Lemma 2.** If \(X\) is a sum of right-handed alternating games with at least one
component in \(L^-\), then \(X \in L^-\).

*Proof.* Left wins trivially playing first on \(X\). Assume, for followers of \(X\), Left wins
playing second on a sum of right-handed games when at least one of them is in \(L^-\).
Let \(G\) be a component of \(X\) in \(L^-\). If Right plays in \(G\) to some \(G^R\) then Left either
has no response and wins immediately, or responds with a good move \(G^{RL} \in L^-\) and
wins by induction on the sum. If Right plays in some other component of
\(X\) then Left either has no response and wins, or can play in that component to

\(^4\)We will find that \(\{0 \mid \cdot\}\) is incomparable with zero in our universe; for this and other reasons,
we chose not to use the normal-play name ‘1’.
bring it back to a right-handed game, and then wins playing second on the sum, by induction, since $G$ is still in $\mathcal{L}^-$.

\begin{proof}
Lemma 3. If $G$ is a single right-handed game in $\mathcal{P}^+$ then $G \equiv 0 \pmod{A}$, and if $G$ is a single left-handed game in $\mathcal{P}^+$ then $G \equiv 0 \pmod{A}$.

$\text{Proof.}$ We prove the first statement, and the second will follow by symmetry. Let $G$ be a right-handed game with normal outcome $\mathcal{P}^+$ and $X$ be a sum of alternating games. We need to show that $o^-(G + X) \geq o^-(X)$. If $o^-(X) = R^-$ then trivially $o^-(G + X) \geq o^-(X)$. It remains to show that if Left can win $X$ going first (second) then she can win $G + X$ going first (second). Assume inductively that these statements hold for all followers of $G + X$ of the form $G' + X'$ with $G' \in \mathcal{P}^+$.

Suppose now that Left can win $X$ going first. If she has no move in $X$ then she has no move in $G + X$ and so wins immediately. Otherwise, she has a move to some $X^L$ from which she wins moving second, and by induction she then wins $G + X^L$ moving second. Thus Left can win $G + X$ moving first.

Suppose Left wins $X$ moving second. Note that Right always a has a move at the outset. If Right moves in $G$ then Left has a (normal-play) winning local response to some $G^{RL}$, which as a right-handed game must be in $\mathcal{P}^+$ and not $\mathcal{L}^+$. Then Left wins $G^{RL} + X$ by induction. If Right moves in $X$ to $X^R$ then either Left wins because she has no move, or she responds with her good second move $X^{RL}$ and by induction wins $G + X^{RL}$ moving second.

We are now set to prove our earlier claim that every one-handed game in $\mathcal{N}^0 \cap \mathcal{P}^+$ is indistinguishable – among all alternating games – from zero.

Theorem 4. If $G$ is any one-handed alternating game in $\mathcal{N}^0 \cap \mathcal{P}^+$ then $G \equiv 0 \pmod{A}$.

$\text{Proof.}$ Assume $G$ is left-handed (the other case is symmetric). By Lemma 3 we already have $G \equiv 0$, and so it suffices to show $G \equiv 0$. Let $X$ be any sum of alternating games. Assume that for all followers $X'$ of $X$, Left can win $G + X'$ going first (second) when she can win $X'$ going first (second).

Suppose Left wins $X$ playing first. If she has no move in $X$, then in $G + X$ she can make a good first misère move in $G \in \mathcal{N}^-$, say $G^L \in \mathcal{L}^-$, to bring the whole position to a sum of right-handed games one of which is in $\mathcal{L}^-$. Lemma 2 then shows Left wins playing first on $G + X$. Otherwise, Left has a move in $X$ to some $X^L$ from which she wins playing second; Left then wins $G + X^L$ playing second, by induction, and so wins $G + X$ playing first.

Suppose Left wins $X$ playing second. Then Left can win $X^R$ playing first for any Right move $X^R$, and so by induction wins $G + X^R$ playing first. Since Right has no choice but to play in $X$ on the sum $G + X$, this shows Left wins playing second on $G + X$.
We want to show that every one-handed alternating game not in $N^{-\land}P^+$ is also equivalent to every other game with the same pair of outcomes. We first establish two instances of domination among left options: Left should move to a position in $L^{-\land}P^+$ over one in $N^{-\land}P^+$, and should choose $N^{-\land}P^+$ over a position in $N^{-\land}R^+$. Naively, Left prefers one option over another if the misère or normal outcome is more favourable for Left and the other outcome is just the same. As with normal-play, if $G = \{G^L \mid G^R\}$, then we can remove dominated options from $G^L$ and $G^R$ to obtain a game equivalent to $G$ under misère-play [9].

**Theorem 5.** If $G \in N^{-\land}R^+$ and $H \in L^{-\land}P^+$ are one-handed alternating games, then $G \leq 0 \leq H \ (\text{mod } A)$.

**Proof.** Note that $G$ and $H$ are right-handed games. Since $H \in P^+$, Lemma 2 gives us $0 \leq H$. For $G \leq 0$, let $X$ be a sum in $A$. Suffices to show that if Right can win $X$ playing first (second) then Right can win $G + X$ playing first (second); assume this is true for all followers $X'$ of $X$. Now suppose Right can win $X$ playing first. If Right has no moves in $X$, he can make a good first misère move in $G$ to $G^R \in R^-$ and win on $G^R + X$ by Lemma 2. Otherwise, Right makes his good misère move in $X$ to $X^R$; since Right wins $X^R$ playing second, he wins $G + X^R$ playing second by induction. This shows Right wins $G + X$ playing first.

If Right wins $X$ playing second then Left has no good first move in $X$ and cannot play the right-handed game $G$. Left must move $X$ to $X^L$, which Right can win playing first, and then by induction Right can win $G + X^L$ playing first. Thus Right wins $G + X$ playing second. \quad \Box

The symmetric result obviously holds as well, and together these give us a ‘partial’ partial order of moves. In particular, Theorem 5 tells us that $C \leq 0 \leq B$ and $B \leq 0 \leq C$. We will be able to establish another chain of domination ($C \leq A \leq B$, $B \leq A \leq C$) after we prove Theorem 6.

**Theorem 6.** If $G$ is a one-handed alternating game, then the following statements are true modulo $A$:

(i) If $G \in R^{-\land}L^+$ then $G \equiv A$.  
(ii) If $G \in L^{-\land}R^+$ then $G \equiv A$.

(iii) If $G \in N^{-\land}L^+$ then $G \equiv B$.  
(iv) If $G \in N^{-\land}R^+$ then $G \equiv B$.

(v) If $G \in R^{-\land}P^+$ then $G \equiv C$.  
(vi) If $G \in L^{-\land}P^+$ then $G \equiv C$.

**Proof.** (i)-(ii): If $G \in R^{-\land}L^+$ then $G$ is left-handed, and since Left has no good first misère move, every Left option must be in $N^{-\land}$ or $R^{-\land}$. But all Left options are right-handed (or zero), and thus cannot be in $R^{-\land}$. So every Left option is in $N^{-\land}$. Similarly, since $G \in L^+$ there is at least one Left option in $P^+$ or $L^+$; but right-handed (or zero) games cannot be in $L^+$, so at least one Left option is in $P^+$. Together this shows there exists a $G^L \equiv 0 \in N^{-\land}P^+$. By Theorem 5, any other
options (necessarily in $\mathcal{N}^{-} \cap \mathcal{R}^{+}$) are dominated by 0, and so $G \equiv \{ 0 \mid \cdot \} \equiv A$. Case (ii) follows by symmetry.

(iii)–(vi): We combine the remaining results and use an inductive proof. The reader can check that $B$ is the smallest one-handed game in $\mathcal{N}^{-} \cap \mathcal{L}^{+}$ (that is, no one-handed game with a shorter game tree is in $\mathcal{N}^{-} \cap \mathcal{L}^{+}$), and $B$, $C$, and $\overline{C}$ are the smallest games in $\mathcal{R}^{-} \cap \mathcal{P}^{+}$, $\mathcal{N}^{-} \cap \mathcal{R}^{+}$, and $\mathcal{L}^{-} \cap \mathcal{P}^{+}$, respectively.

We now proceed by induction on the followers. Let $G$ and $H$ be games in $\mathcal{R}^{-} \cap \mathcal{P}^{+}$ and $\mathcal{N}^{-} \cap \mathcal{L}^{+}$ (note that $G$ and $H$ are left-handed; the right-handed results follow similarly). As argued in (i)–(ii), every Left option of $G$ must be in $\mathcal{N}^{-}$ because Left has no good first misère move in $G \in \mathcal{R}^{-}$. Additionally, $G \in \mathcal{P}^{+}$ forces every Left option to be in $\mathcal{R}^{+}$ or $\mathcal{N}^{+}$; but one-handed games cannot be in $\mathcal{N}^{+}$, so every option is in $\mathcal{N}^{-} \cap \mathcal{R}^{+}$. By induction each of these options is indistinguishable from $\overline{B}$. Thus $G \equiv \{ \overline{B} \mid \cdot \} = C$.

For $H$ there is a bit more to show. Since $H \in \mathcal{N}^{-}$, Left has a good misère move to a right-handed game in $\mathcal{L}^{-}$; since $H \in \mathcal{L}^{+}$, Left has a good normal-play move to a right-handed game in $\mathcal{P}^{+}$. This shows that Left either has a move to $\mathcal{L}^{-} \cap \mathcal{P}^{+}$ ($\equiv \overline{C}$, by induction) or has moves to both $\mathcal{L}^{-} \cap \mathcal{R}^{+}$ ($\equiv \overline{A}$ by (i)) and $\mathcal{N}^{-} \cap \mathcal{P}^{+}$ ($\equiv 0$). If there are any moves to $\mathcal{N}^{-} \cap \mathcal{R}^{+}$, they are dominated by 0 or $\overline{C}$, by Theorem 5. Using this and the domination of $\overline{C}$ over 0, we reduce the possibilities for $H$ to

$$H \equiv \{ \overline{A}, 0 \mid \cdot \}, \{ \overline{C} \mid \cdot \}, \text{or } \{ \overline{A}, \overline{C} \mid \cdot \}.$$

In the first case we have $H \equiv B$ immediately. It remains to show that $\{ \overline{C} \mid \cdot \} \equiv B$ and $\{ \overline{A}, \overline{C} \mid \cdot \} \equiv B$.

Claim 1: $\{ \overline{C} \mid \cdot \} \equiv B$. Let $J = \{ \overline{C} \mid \cdot \}$. We need to show $\omega^{-}(B + X) = \omega^{-}(J + X)$ for all sums $X$ in $A$. Assume true for all followers of $X$. If Right wins playing first in $B + X$ then he must win second from some $B + X^R$, which by induction has the same outcome as $J + X^R$, and then Right wins first from $J + X$ with the same move in $X$. If Left wins playing second in $B + X$ the same argument shows that Left wins playing second in $J + X$, since for every $X^R$, $\omega^{-}(B + X^R) = \omega^{-}(J + X^R)$.

If Right wins playing second on $B + X$, and Left’s first move in $J + X$ is to some $J + X^L$, then since Right can win playing first on $B + X^L$ he can win playing first from $J + X^L$, by induction. Otherwise, Left’s first move in $J + X$ is to $\overline{C} + X$, and Right should respond with $B + X$, from which he wins playing second.

Finally, suppose Left wins $B + X$ playing first. If her good move is in $X$ then as above Left wins $J + X$ with the same first move. If Left’s good first move is to $B^L + X$ then Left must be able to win playing second from any position $B^L + X^{RL\ldots RL}$ obtained from optimal Left play (including $B^L + X$). Now, in $J + X$, Left should move first to $\overline{C} + X$ and play her original strategy in $X$. When Right chooses to play in $\overline{C}$, to $B + X^{RL\ldots RL}$, for some (not necessarily proper) follower of $X$, Left plays to $B^L + X^{RL\ldots RL}$ and wins from there playing second.
Claim 2: \( \{ \overline{A}, \overline{C} \mid \cdot \} \equiv B \). Let \( K = \{ \overline{A}, \overline{C} \mid \cdot \} \). Note that when at least one left option already exists, introducing another left option cannot create a position worse for Left; Left simply ‘ties her hand’ and ignores the extra option. In this way we see that \( K \geq J \equiv B \). It remains to show that \( K \leq B \); i.e., that Right wins \( K + X \)
whenever he wins \( B + X \).

If Right wins \( B + X \) playing first then he wins \( K + X \) playing first, by induction, as in claim 1. Suppose Right wins \( B + X \) playing second. If Left moves first in \( K + X \) to \( \overline{C} + X \) then Right wins by moving \( \overline{C} \) to \( B \) and then playing second on \( B + X \). If Left moves first in \( K + X \) to \( \overline{A} + X \) then Right can win first from that position, since \( \overline{A} + X \) is a possible first Left move from \( B + X \).

Theorem 3.5 is a powerful result. Since the equivalences hold modulo all alternating games, and since every alternating game becomes one-handed after the first move, these equivalences are pertinent to future research in the universe \( A \). As promised, Theorem 3.5 also allows us to prove the missing chain of domination; we can then combine the results of Theorems 5 and 7 (below) to obtain the partial orders illustrated in Figure 3. As indicated by the figure, the game 0 is incomparable or ‘fuzzy’ with both \( A \) and \( \overline{A} \). To see this, note that \( 0 + * \in P^- \) while \( A + * \in N^- \).

**Theorem 7.** If \( G \in N^- \cap R^+, H \in L^- \cap P^+, \) and \( K \in L^- \cap R^+ \) are one-handed alternating games, then \( G \leq K \leq H \) modulo \( A \).

**Proof.** By Theorem 6 we need only show \( B \leq \overline{A} \leq C \). As in the proof of Claim 2 above, we immediately have \( B \leq \overline{A} \), since Right can ‘tie his hand’ in \( B = \{ \cdot \mid 0, A \} \) and pretend he is playing with the position \( \overline{A} = \{ \cdot \mid 0 \} \); that is, \( B \) is at least as good as \( \overline{A} \) for Right.

To see that \( \overline{A} \leq C \), let \( X \) be any sum of alternating games, and suppose Left wins \( \overline{A} + X \) playing first. Then we know Left must be able to win playing second from \( \overline{A} + X^{LR\ldots L} \), for any \( X^{LR\ldots L} \) obtained from \( X \) under optimal play. In \( \overline{C} + X \), Left plays as usual until Right moves from \( \overline{C} + X^{LR\ldots L} \) to \( B + X^{LR\ldots L} \); Left then moves to \( \overline{A} + X^{LR\ldots L} \) and wins playing second from there. A similar argument shows Left wins \( \overline{C} + X \) playing second whenever she wins \( \overline{A} + X \) playing second, and so \( \overline{A} \leq C \).

We have nearly finished our analysis of one-handed games in the alternating universe. The final result of this section addresses one of the main difficulties in misère game theory: the lack of inverses under disjunctive sum. As mentioned in the introduction, we do not generally have \( \overline{G} = -G \) in misère games as we do for normal games. Luckily for us, one-handed alternating games do possess this very convenient property.

**Theorem 8.** If \( G \) is a one-handed alternating game then \( G + \overline{C} \equiv 0 \) (mod \( A \)).
Proof. This is trivially true for games in $\mathcal{N}^+ \cap \mathcal{P}^+$.

For any other (say left-handed) game $G$, we have $G$ is equivalent (modulo $\mathcal{A}$) to $A, B,$ or $C$. Thus, it suffices to show $A + A \equiv 0, B + B \equiv 0,$ and $C + C \equiv 0$. Let $X$ be any sum of alternating games and suppose without loss of generality that Left wins $X$. Then in $A + A + X,$ Left plays his winning strategy in $X$, responding in $A + A$ (and thereby bringing it to zero) only if Right plays there. If this happens before Left runs out of move in $X$ then play resumes in $X$ and Left wins as usual. Otherwise Right does not play in $A + A$ before Left wins $X$, at which point Left plays in $A$, bringing the game to a sum of right-handed components including $A \in \mathcal{L}^-$. Left then wins by Lemma 2. Notice this argument works whether Left wins $X$ playing first or playing second or both.

For $B + B + X$, the argument works as above; if Right plays in $B$ then Left copies her in $B$, bringing those components to $0 + 0$ or $A + A \equiv 0$, and resumes play in $X$. If Left runs out of moves in $X$ before this happens, she plays in $B$ to $A$ and wins by Lemma 2. Finally, in $C + C + X$, if Right plays in $C$ then Left copies in $C$ to get $B + B \equiv 0$. If Right does not play in $C$ then after Left runs out of moves in $X$ he plays in $C$ to $B$. The game is now a sum of right-handed positions including $B \in \mathcal{L}^-$, so once again Left wins by Lemma 2.

Figure 3: The partial orders (modulo $\mathcal{A}$) given by Theorems 5 and 7.

Theorem 8 shows that disjunctive sum forms the set of one-handed alternating games into a group, $\mathcal{G}$ is in fact $-G$. We use the notations interchangeably for the remainder of the paper and write $-kG$ to represent $k$ copies of the game $-G$.

Together, Theorems 6 and 8 show that any sum of one-handed games in the alternating universe $\mathcal{A}$ can be written as $aA + bB + cC$ for integers $a, b, c$. In Section 3 we determine the outcome of such a sum and in Section 4 we use this information to determine the misère monoid of $\mathcal{O}$. 


3. Outcomes

If $G$ is a sum of one handed games in $\mathcal{A}$ then $G \equiv aA + bB + cC$ for integers $a, b, c$. We can therefore represent any such sum as an ordered triple $(a, b, c)$. Figure 4 illustrates the possible Left and Right options from $G$ (with the conditions for each indicated below the horizontal line), and Theorem 9 establishes the misère outcome of $G$.

$$
\begin{array}{c|c|c|c|c|c|c}
(a-1,b,c) & (a,b-1,c) & (a,b-1,c-1) & (a+1,b,c) & (a,b+1,c) & (a,b+1,c+1) \\
\hline
\text{if } a>0, & \text{if } b>0, & \text{if } c>0, & \text{if } a<0, & \text{if } b<0, & \text{if } c<0, \\
A \rightarrow 0 & B \rightarrow 0, \mathcal{A} & C \rightarrow \mathcal{B} & \mathcal{X} \rightarrow 0 & \mathcal{B} \rightarrow 0, \mathcal{A} & \mathcal{B} \rightarrow \mathcal{B}
\end{array}
$$

Figure 4: Left and Right options from $G = aA + bB + cC$.

Theorem 9. Let $G = aA + bB + cC$. Then

$$
o^−(G) = \begin{cases} 
\mathcal{L}^−, & \text{if } a + c < 0, \\
\mathcal{N}^−, & \text{if } a + c = 0, \\
\mathcal{R}^−, & \text{if } a + c > 0.
\end{cases}
$$

Proof. The zero game is a next-player win under misère rules and has $a + c = 0$. Assume the outcomes above hold for all followers of $G = (a, b, c)$.

If $a + c < 0$ then there exists at least one copy of $\mathcal{A}$ or $\mathcal{C}$ in the sum, so Right has a move going first to $(a + 1, b, c)$ or $(a, b + 1, c + 1)$. At best then, Right can increase the value of $a + c$ to zero, giving Left a next-win position by induction. If Left plays first she can play to either $(a - 1, b, c)$ or $(a, b - 1, c - 1)$, and so can guarantee $a + c$ remains negative and by induction leaves a Left-win position. Since Left wins playing first or second, $G \in \mathcal{L}^−$ when $a + c < 0$. A symmetric argument shows $G \in \mathcal{R}^−$ when $a + c > 0$.

If $a + c = 0$ then Left going first either has no moves, and wins immediately, or has a move in a copy of $A, B$, or $C$. From Figure 4 we see Left can always decrease either $a$ or $c$ by 1, thereby moving to $a + c < 0$ and winning by induction. Right wins playing first by symmetry, and so $G \in \mathcal{N}^−$ when $a + c = 0$.

This theorem will be very useful in analyzing the two-handed universe. For the present paper, it shows us that the equivalence classes among one-handed games collapse even further when the universe is restricted to $\mathcal{O}$. Notice that the integer $b$ does not influence the outcome of $G = aA + bB + cC$. Thus, for $X$ a sum of only one-handed alternating games, we have $o^−(X) = o^−(B + X)$. This gives the following corollaries.
Corollary 10. If $O$ is the closure of one-handed alternating games, then $B \equiv 0 \pmod{O}$ and $C \equiv A \pmod{O}$.

Corollary 11. If $G$ is a sum of one-handed games then $G \equiv aA \pmod{O}$ for some integer $a$.

Note that we do have to restrict the universe to $O$ to get these results. In the two-handed universe, $B$ is distinguished from 0 with the game $* = \{0 \mid 0\}$: Left can win $B + *$ going first but loses playing first on $*$.

The last thing to note in our analysis of $O$ is that the positions $aA$ and $a’A$ are equivalent if and only if $a = a’$. If $a \neq a’$ then the games are distinguished by $-aA$, since $aA + (-aA)$ is in $N^-$ while $a’A + (-aA)$ is in $L^-$ or $R^-.$

4. The Mise`ere Monoid

The previous section gives us everything we need to describe the one-handed alternating universe $O$. Let $\alpha$ represent the equivalence class of the game $A$ and let $e$ represent the equivalence class of the game 0. The mis`ere monoid of $O$ is given by

$$
\mathcal{M}_O = \langle e, \alpha, \alpha^{-1} \mid \alpha \cdot \alpha^{-1} = e \rangle,
$$

$$
N^- = \{0\}, \mathcal{P}^- = \emptyset, \mathcal{R}^- = \{a^a \mid a \in \mathbb{N}\}, \mathcal{L}^- = \{\alpha^{-a} \mid a \in \mathbb{N}\}.
$$

As noted earlier, this monoid is actually a group; the mapping $aA \mapsto a$ for $a \in \mathbb{Z}$ shows $\mathcal{M}_O \cong (\mathbb{Z}, +)$.

References