ON THE SUM OF RECIPROCAL GENERALIZED FIBONACCI NUMBERS

Sarah H. Holliday
Dept. of Mathematics, Southern Polytechnic State University, Marietta, Georgia
shollida@spsu.edu

Takao Komatsu1
Graduate School of Science and Technology, Hirosaki University, Hirosaki, Japan
komatsu@cc.hirosaki-u.ac.jp

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Abstract
The Fibonacci Zeta functions are defined by \( \zeta_F(s) = \sum_{k=1}^{\infty} F_k^{-s} \). Several aspects of the function have been studied. In this article we generalize the results by Ohutsuka and Nakamura, who treated the partial infinite sum \( \sum_{k=n}^{\infty} F_k^{-s} \) for all positive integers \( n \).

1. Introduction
The so-called Fibonacci and Lucas Zeta functions, defined by

\[
\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad \text{and} \quad \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s},
\]

respectively, have been considered in several different ways. In [8] the analytic continuation of these series is discussed. In [2] it is shown that the numbers \( \zeta_F(2), \zeta_F(4), \zeta_F(6) \) (respectively, \( \zeta_L(2), \zeta_L(4), \zeta_L(6) \)) are algebraically independent, and that each of \( \zeta_F(2s) \) (respectively, \( \zeta_L(2s) \)) \( (s = 4, 5, 6, \ldots) \) can be written as a rational (respectively, algebraic) function of these three numbers over \( \mathbb{Q} \). Similar results are obtained in [2] for the alternating sums

\[
\zeta_F^*(2s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^{2s}} \quad \text{and} \quad \zeta_L^*(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^{2s} L_n^s} \quad (s = 1, 2, 3, \ldots).
\]

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From the main theorem in [4] it follows that for any positive distinct integers \( s_1, s_2, s_3 \) the numbers \( \zeta_F(2s_1), \zeta_F(2s_2), \) and \( \zeta_F(2s_3) \) are algebraically independent if and only if at least one of \( s_1, s_2, s_3 \) is even. Other types of algebraic independence, including the functions

\[
\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^3}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n}^3}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^3}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^3},
\]

are discussed in [6]. In [5] Fibonacci zeta functions and Lucas zeta functions including

\[ \zeta_F(1), \zeta_F(2), \zeta_F(3), \zeta_F^*(1), \zeta_L(1), \zeta_L(2), \zeta_L^*(1) \]

are expanded as non-regular continued fractions whose components are Fibonacci or Lucas numbers.

In [9] the partial infinite sums of reciprocal Fibonacci numbers were studied. In this paper we shall generalize their results, given in Propositions 1 and 2 below. Here, \([ \cdot ]\) denotes the floor function.

**Proposition 1.** We have

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right] = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}
\]

**Proposition 2.** We have

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right] = \begin{cases} F_{n-1} F_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1} F_n & \text{if } n \text{ is odd and } n \geq 1. \end{cases}
\]

2. **Main Results**

Let \( a \) be a positive integer. Let \( \{G_n\} \) be a general Fibonacci sequence defined by \( G_{k+2} = aG_{k+1} + G_k \) (\( k \geq 0 \)) with \( G_0 = 0 \) and \( G_1 = 1 \).

**Theorem 3.** We have

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right] = \begin{cases} G_n - G_{n-1} & \text{if } n \text{ is even and } n \geq 2; \\ G_n - G_{n-1} - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}
\]

**Theorem 4.** We have

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} \right] = \begin{cases} aG_{n-1} G_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ aG_{n-1} G_n & \text{if } n \text{ is odd and } n \geq 1. \end{cases}
\]
We need some identities in order to prove Theorems 1 and 2.

**Lemma 5.** For \( n \geq 1 \), we have

1. \( G_n^2 - G_{n-1}G_{n+1} = (-1)^{n-1} \)
2. \( G_{n-1}G_{n+1} - G_nG_{n+2} = (-1)^n(a^2 + 1) \)
3. \( G_nG_{n+2} + G_{n-1}G_{n+1} = G_{2n+1} \)
4. \( G_{n+1}G_{n+2} - G_{n-1}G_n = aG_{2n+1} \).

**Proof.** Every proof is done by induction and omitted. \( \square \)

**Proof of Theorem 3.** Using Lemma 5 (1), for \( n \geq 1 \) we have

\[
\frac{1}{G_n - G_{n-1}} - \frac{1}{G_n} = \frac{1}{G_n + G_n - G_{n+1}} - \frac{1}{G_n + G_n - G_{n+1}} = \frac{G_n^2 - G_{n-1}G_{n+1}}{(G_n - G_{n-1})(G_n - G_{n-1})} = \frac{(-1)^n(G_{n+1} - G_{n-1})}{G_nG_n(G_n + G_n - G_{n+1})},
\]

(1)

If \( n \) is even with \( n \geq 2 \), since the right-hand side of the identity (1) is positive, we get

\[
\frac{1}{G_n - G_{n-1}} > \frac{1}{G_n} + \frac{1}{G_n + 1} + \frac{1}{G_n + 2 - G_{n+1}}.
\]

(2)

By applying inequality (2) repeatedly we have

\[
\frac{1}{G_n - G_{n-1}} > \frac{1}{G_n} + \frac{1}{G_n + 1} + \frac{1}{G_n + 2 - G_{n+1}}
\]

\[
> \frac{1}{G_n} + \frac{1}{G_n + 1} + \frac{1}{G_n + 2} + \frac{1}{G_n + 3} + \frac{1}{G_n + 4 - G_{n+3}}
\]

\[
> \frac{1}{G_n} + \frac{1}{G_n + 1} + \frac{1}{G_n + 2} + \frac{1}{G_n + 3} + \frac{1}{G_n + 4} + \frac{1}{G_n + 5} + \cdots.
\]

Thus, we obtain

\[
\sum_{k=n}^{\infty} \frac{1}{G_k} < \frac{1}{G_n - G_{n-1}}.
\]

(3)
In a similar way, if \( n \) is odd with \( n \geq 1 \), then
\[
\sum_{k=n}^{\infty} \frac{1}{G_k} > \frac{1}{G_n - G_{n-1}}.
\]
(4)

On the other hand, if \( n \) is even with \( n \geq 2 \), then by Lemma 5, parts (1) and (4)
\[
\frac{1}{G_n - G_{n-1} + 1} - \frac{1}{G_n} - \frac{1}{G_n + 2 - G_{n+1} + 1}
= -\frac{2(-1)^{n-1} + (-1)^n G_{n+2} + (-1)^n G_{n-1} + aG_{2n+1} + G_n + G_{n+1}}{G_n G_{n+1} (G_n - G_{n-1} + 1) (G_{n+2} - G_{n+1} + 1)}
= -\frac{(aG_{2n+1} - G_{n+2}) + (G_{n-1} + G_n + G_{n+1} - 2)}{G_n G_{n+1} (G_n - G_{n-1} + 1) (G_{n+2} - G_{n+1} + 1)} < 0.
\]

Hence, by applying the inequality
\[
\frac{1}{G_n - G_{n-1} + 1} < \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2} - G_{n+1} + 1}
\]
repeatedly, we obtain
\[
\frac{1}{G_n - G_{n-1} + 1} < \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2} - G_{n+1} + 1}
< \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2}} + \frac{1}{G_{n+3} - G_{n+3} + 1}
< \frac{1}{G_n} + \frac{1}{G_{n+1}} + \frac{1}{G_{n+2}} + \frac{1}{G_{n+3}} + \frac{1}{G_{n+4}} + \frac{1}{G_{n+5}} + \cdots.
\]

Thus,
\[
\frac{1}{G_n - G_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{1}{G_k}.
\]

Together with (3) we have
\[
\frac{1}{G_n - G_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{1}{G_k} < \frac{1}{G_n - G_{n-1}},
\]
so
\[
\left[ \sum_{k=n}^{\infty} \frac{1}{G_k} \right]^{-1} = G_n - G_{n-1}.
\]
In a similar manner, if \( n \) is odd with \( n \geq 1 \), then

\[
\frac{1}{G_n - G_{n-1} - 1} - \frac{1}{G_n} - \frac{1}{G_{n+1} - G_{n+2} - G_{n+1} - 1} = 2(-1)^{n-1} + (-1)^nG_{n+2} + (-1)^nG_{n-1} + aG_{2n+1} - G_n - G_{n+1}
\]

\[
= \frac{aG_{n+1}(G_{n+1} - 1) + G_n(aG_n - a - 2) + 2}{G_nG_{n+1}(G_n - G_{n-1} - 1)(G_n + 2 - G_{n+1} - 1)} \geq 0,
\]

where the equality holds only for \( n = a = 1 \). Hence,

\[
\frac{1}{G_n - G_{n-1} - 1} > \sum_{k=n}^{\infty} \frac{1}{G_k}.
\]

Together with (4) we have

\[
\frac{1}{G_n - G_{n-1} - 1} < \sum_{k=n}^{\infty} \frac{1}{G_k} < \frac{1}{G_n - G_{n-1} - 1},
\]

so

\[
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right] = G_n - G_{n-1} - 1.
\]

\[\square\]

Proof of Theorem 4. By Lemma 5(1)

\[
\frac{1}{aG_{n-1}G_n - 1} - \frac{1}{G_n} - \frac{1}{aG_nG_{n+1} - 1} = \frac{a(G_nG_{n+1} - G_{n-1}G_n)}{(aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)} - \frac{1}{G_n^2}
\]

\[
= \frac{a^2G_n^4 - (aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)}{G_n^2(aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)}
\]

\[
= \frac{a^2G_n^4(-1)^{n-1} + aG_n(G_{n-1} + G_{n+1}) - 1}{G_n^2(aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)}
\]

\[
\geq \frac{2aG_{n-1}G_n - 1}{G_n^2(aG_{n-1}G_n - 1)(aG_nG_{n+1} - 1)}
\]

\[
> 0.
\]
Therefore,
\[
\frac{1}{aG_{n-1}G_n - 1} > \frac{1}{G_n^2} + \frac{1}{aG_nG_{n+1} - 1} > \frac{1}{G_n^2} + \frac{1}{G_{n+1}^2} + \frac{1}{aG_{n+1}G_{n+2} - 1} > \ldots
\]

Thus, we have
\[
\frac{1}{aG_{n-1}G_n - 1} > \sum_{k=n}^{\infty} \frac{1}{G_k^2}.
\] (5)

In a similar way,
\[
\frac{1}{aG_{n-1}G_n + 1} - \frac{1}{G_n^2} - \frac{1}{aG_nG_{n+1} + 1} \leq -\frac{2aG_{n-1}G_n + 1}{G_n(aG_{n-1}G_n + 1)(aG_nG_{n+1} + 1)} < 0.
\]

Thus, we have
\[
\frac{1}{aG_{n-1}G_n + 1} < \sum_{k=n}^{\infty} \frac{1}{G_k^2}.
\] (6)

On the other hand, by Lemma 5(1) and (3),
\[
\frac{1}{aG_{n-1}G_n} - \frac{1}{G_n} - \frac{1}{G_{n+1}} - \frac{1}{aG_{n+1}G_{n+2}}
\] 
\[
= \frac{G_{n-2}}{aG_{n-1}G_n^2} - \frac{G_{n+3}}{aG_{n+1}G_{n+2}}
\] 
\[
= \frac{G_{n-2}G_{n+1}^2 G_{n+2} - G_{n-1}G_n^2 G_{n+3}}{aG_{n-1}G_{n+1}G_{n+2}^2}
\] 
\[
= \frac{a^2(G_n^2 - G_{n-1}G_{n+1})(G_nG_{n+2} + G_{n-1}G_{n+1})}{aG_{n-1}G_{n+1}G_{n+2}^2}
\] 
\[
= \frac{a(-1)^n - G_{2n+1}}{G_{n-1}G_{n+1}^2 G_{n+2}}.
\]
If $n$ is even with $n \geq 2$, then
\[
\frac{1}{aG_{n-1}G_n} < \frac{1}{G_n^2} + \frac{1}{G_{n+1}^2} + \frac{1}{aG_{n-1}G_{n+2}} < \frac{1}{G_{n+1}^2} + \frac{1}{G_{n+2}^2} + \frac{1}{G_{n+3}^2} + \frac{1}{aG_{n+3}G_{n+4}} < \ldots < \frac{1}{G_n^2} + \frac{1}{G_{n+1}^2} + \frac{1}{G_{n+2}^2} + \frac{1}{G_{n+3}^2} + \frac{1}{G_{n+4}^2} + \ldots.
\]
Hence, we have
\[
\sum_{k=n}^{\infty} \frac{1}{G_k^2} > \frac{1}{aG_{n-1}G_n}. \quad (7)
\]
Similarly, if $n$ is odd with $n \geq 1$, then
\[
\sum_{k=n}^{\infty} \frac{1}{G_k^2} < \frac{1}{aG_{n-1}G_n}. \quad (8)
\]
If $n$ is even with $n \geq 2$, then by equations (5) and (7) we obtain
\[
aG_{n-1}G_n - 1 < \left( \sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} < aG_{n-1}G_n.
\]
Thus,
\[
\left( \sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} = aG_{n-1}G_n - 1.
\]
If $n$ is odd with $n \geq 1$, then by equations (6) and (8) we obtain
\[
aG_{n-1}G_n < \left( \sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} < aG_{n-1}G_n + 1.
\]
Thus,
\[
\left( \sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} = aG_{n-1}G_n.
\]
\[\square\]

The following results are proved in similar manners. Such reciprocal sums of Fibonacci-type numbers have been studied by several authors (e.g. [1], [3], [6], [11]).
Theorem 6. We have

\[
\begin{align*}
(1) & \quad \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k}} \right)^{-1} = G_{2n} - G_{2n-2} - 1 \quad (n \geq 1) \\
(2) & \quad \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k-1}} \right)^{-1} = G_{2n-1} - G_{2n-3} \quad (n \geq 2) \\
(3) & \quad \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k-1}G_{2k+1}} \right)^{-1} = G_{4n-1} - G_{4n-3} \quad (n \geq 1) \\
(4) & \quad \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k}G_{2k+2}} \right)^{-1} = G_{4n+1} - G_{4n-1} - 1 \quad (n \geq 1) \\
(5) & \quad \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k}^2} \right)^{-1} = G_{4n-1} - G_{4n-3} - 1 \quad (n \geq 1) \\
(6) & \quad \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k-1}^2} \right)^{-1} = G_{4n-3} - G_{4n-5} \quad (n \geq 2) \\
(7) & \quad \left( \sum_{k=n}^{\infty} \frac{1}{G_{2k-1}G_{2k}} \right)^{-1} = G_{4n-2} - G_{4n-4} \quad (n \geq 1).
\end{align*}
\]

3. Generalized Fibonacci Numbers

Let \( c \) be a non-negative integer. Let \( \{H_n\} \) be a generalized Fibonacci sequence defined by \( H_{k+2} = H_{k+1} + H_k \) \((k \geq 0)\) with \( H_0 = c \) and \( H_1 = 1 \).

Note that \( H_n = F_{n+1} \) if \( c = 1 \), and \( H_n = L_n \) (Lucas numbers) if \( c = 2 \) ([7, Corollary 5.5 (5.14)])).

The sequence \( H_n \) can be defined also as the total number of matchings in the connected planar graph on \( n \) vertices with \( n - 2 + c \) total edges, of which \( c - 1 \) edges are between one pair of vertices. The \( c = 1 \) and \( c = 2 \) cases are stated in [10, A45 and A204], and the proof for \( c > 2 \) is an inductive counting argument. An similar result for the Fibonacci type sequence \( G_{k+2} = aG_{k+1} + G_k \), \( G_0 = 0 \), \( G_1 = 1 \) can be generated by counting the total matchings in a path (as defined in [12] on \( k - 1 \) vertices with \( a \) loops at each vertex.)
Theorem 7. We have

$$
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{H_k} \right)^{-1} \right] = \begin{cases} 
H_{n-2} - 1 & \text{if } n \text{ is even and } n \geq n_0; \\
H_{n-2} & \text{if } n \text{ is odd and } n \geq n_1.
\end{cases}
$$

Remark. $n_0$ and $n_1$ are determined depending only on the value of $c$. For example, if $H_k = L_k$ (Lucas number) or $c = 2$, then $n_0 = 2$ and $n_1 = 3$.

Precisely speaking, $n_0 = 2$ if $c = 1, 2$; $n_0 = 4$ if $c \leq 4$; $n_0 = 6$ if $c \leq 10$; $n_0 = 8$ if $c \leq 26$; $n_0 = 10$ if $c \leq 68$; $n_0 = 12$ if $c \leq 178$; $n_0 = 14$ if $c \leq 466$; $n_0 = 16$ if $c \leq 1220$; $n_0 = 18$ if $c \leq 3194$; $n_0 = 20$ if $c \leq 8362$.

Similarly, $n_1 = 1$ if $c = 1$; $n_1 = 3$ if $c = 2$; $n_1 = 5$ if $c \leq 6$; $n_1 = 7$ if $c \leq 16$; $n_1 = 9$ if $c \leq 42$; $n_1 = 11$ if $c \leq 110$; $n_1 = 13$ if $c \leq 288$; $n_1 = 15$ if $c \leq 754$; $n_1 = 17$ if $c \leq 1974$; $n_1 = 19$ if $c \leq 5168$.

Theorem 8. We have

$$
\left[ \left( \sum_{k=n}^{\infty} \frac{1}{H_k^2} \right)^{-1} \right] = \begin{cases} 
H_{n-1}H_n + g(c) - 1 & \text{if } n \text{ is even and } n \geq n_2; \\
H_{n-1}H_n - g(c) & \text{if } n \text{ is odd and } n \geq n_3,
\end{cases}
$$

where

$$
g(c) = \begin{cases} 
c(c+1) & \text{if } c \equiv 0, 2 \pmod{3}; \\
\frac{3}{c(c+1)+1} & \text{if } c \equiv 1 \pmod{3}.
\end{cases}
$$

Remark. Note that $g(c)$ is an integer. If $H_k = L_k$, then we take $n_2 = 2$ and $n_3 = 1$. Precisely speaking, we can determine $n_2$ and $n_3$ as follows:

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We need some lemmata in order to prove Theorems 7 and 8. Every proof of the lemmata is done by induction and omitted.

Lemma 9. For $n \geq 1$, $H_n = cF_{n-1} + F_n$.

Lemma 10. We have

1. $H_n^2 - H_{n-1}H_{n+1} = H_nH_{n+1} - H_{n-1}H_{n+2} = (-1)^n(c^2 + c - 1)$
2. $H_{n-1}H_{n+1} - H_{n-2}H_{n+2} = (-1)^{n-1}(c^2 + c - 1)$
3. $H_{n+4}H_n - H_{n+2}H_{n-2} = H_{n+1}(H_{n+3} - H_{n-1})$
(4) \(H_{n+1}H_{n+2} - H_{n-1}H_n = H_n^2 + H_{n+1}^2 = cH_{2n} + H_{2n+1}\).

**Proof of Theorem 7.** By Lemma 10 (2)

\[
\frac{1}{H_{n-2}} - 2 \frac{1}{H_{n+1}} - \frac{1}{H_{n+1}} = \frac{(H_{n-2}H_{n+1} - H_{n-2}(H_n + H_{n+1})}{H_{n-2}H_{n+1}H_{n+1}}
\]

\[
= \frac{H_{n-1}H_{n+1} - H_{n-2}H_{n+2}}{H_{n-2}H_{n+1}H_{n+1}}
\]

\[
= \frac{(-1)^{n-1}(c^2 + c - 1)}{H_{n-2}H_{n+1}}.
\]

Hence, if \(c \geq 1\) and \(n\) is even, then by

\[
\frac{1}{H_{n-2}} < \frac{1}{H_n} + \frac{1}{H_{n+1}} + \frac{1}{H_n}
\]

\[
< \frac{1}{H_n} + \frac{1}{H_{n+1}} + \frac{1}{H_{n+2}} + \frac{1}{H_{n+3}} + \frac{1}{H_{n+2}}
\]

\[
< \frac{1}{H_n} + \frac{1}{H_{n+1}} + \frac{1}{H_{n+2}} + \frac{1}{H_{n+3}} + \frac{1}{H_{n+4}} + \frac{1}{H_{n+5}} + \cdots,
\]

we have

\[
\frac{1}{H_{n-2}} < \sum_{k=n}^{\infty} \frac{1}{H_k}.
\]

(9)

In a similar manner, if \(c \geq 1\) and \(n\) is odd, then

\[
\frac{1}{H_{n-2}} > \sum_{k=n}^{\infty} \frac{1}{H_k}.
\]

(10)

On the other hand, if \(n\) is even, then by Lemma 10 (2)

\[
\frac{1}{H_{n-2}} - 1 \frac{1}{H_{n-1}} - \frac{1}{H_n - 1}
\]

\[
= \frac{(-1)^{n-1}(c^2 + c - 1)H_n + H_{n+2}(H_{n-2} + H_n - 1)}{H_nH_{n+1}(H_{n-2} - 1)(H_n - 1)}
\]

\[
= \frac{-2(c^2 + c - 1)H_n}{H_nH_{n+1}(H_{n-2} - 1)(H_n - 1)}.
\]

The numerator is positive if \(n\) is large enough for a fixed \(c\). For example, one can take \(n\) so that \(H_{n+2} > 2(c^2 + c - 1)\) since \(H_n\) is monotone increasing for \(n\). Exactly
speaking, if \( c = 1 \), then the right-hand side of (12) is positive for \( n \geq 2 \). If \( 2 \leq c \leq 4 \), then \( n \geq 4 \). If \( 5 \leq c \leq 9 \), then \( n \geq 6 \). If \( 10 \leq c \leq 24 \), then \( n \geq 8 \). If \( 25 \leq c \leq 62 \), then \( n \geq 10 \). If \( 63 \leq c \leq 161 \), then \( n \geq 12 \). If \( 162 \leq c \leq 422 \), then \( n \geq 14 \). If \( 423 \leq c \leq 1104 \), then \( n \geq 16 \).

If \( n \) is odd, then

\[
\frac{1}{H_{n-2}+1} - \frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_n+1} = (-1)^{n-1} \frac{2(c^2 + c - 1)H_n - H_{n+2}(H_{n-2} + H_n + 1)}{H_nH_{n+1}(H_{n-2} + 1)(H_n + 1)}
\]

The numerator is negative if \( n \) is large enough for a fixed \( c \). For example, if \( c = 1 \), then the right-hand side of (14) is negative for \( n \geq 1 \). If \( c = 2 \), then \( n \geq 3 \). If \( 3 \leq c \leq 6 \), then \( n \geq 5 \). If \( 7 \leq c \leq 15 \), then \( n \geq 7 \). If \( 16 \leq c \leq 38 \), then \( n \geq 9 \). If \( 39 \leq c \leq 100 \), then \( n \geq 11 \). If \( 101 \leq c \leq 261 \), then \( n \geq 13 \). If \( 262 \leq c \leq 682 \), then \( n \geq 15 \).

When \( n \) is even, repeating the inequality

\[
\frac{1}{H_{n-2}-1} - \frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_n-1} > 0,
\]

we have

\[
\frac{1}{H_{n-2}-1} > \sum_{k=n}^{\infty} \frac{1}{H_k}.
\]

Together with (9), we obtain

\[
\left( \sum_{k=n}^{\infty} \frac{1}{H_k} \right)^{-1} = H_{n-2} - 1.
\]

When \( n \) is odd, repeating the inequality

\[
\frac{1}{H_{n-2}+1} - \frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_n+1} < 0,
\]

we have

\[
\frac{1}{H_{n-2}+1} < \sum_{k=n}^{\infty} \frac{1}{H_k}.
\]

Together with (10), we obtain

\[
\left( \sum_{k=n}^{\infty} \frac{1}{H_k} \right)^{-1} = H_{n-2}.
\]
**Proof of Theorem 8.** By Lemma 10 (1)

\[
\frac{1}{H_{n-1}H_n + (-1)^n g(c) - 1} - \frac{1}{H_n^2} - \frac{1}{H_nH_{n+1} + (-1)^{n+1} g(c) - 1} = \frac{H_n^2 + (-1)^{n+1} 2g(c)}{(H_{n-1}H_n + (-1)^n g(c) - 1)(H_nH_{n+1} + (-1)^{n+1} g(c) - 1)} - \frac{1}{H_n^2} \\
= \frac{(-1)^n (c^2 + c - 1 - 3g(c))H_n^2 + (g(c))^2 + H_n(H_n+1 + H_{n-1}) - 1}{H_n^2(H_{n-1}H_n + (-1)^n g(c) - 1)(H_nH_{n+1} + (-1)^{n+1} g(c) - 1)}.
\]

Suppose that \( n \) is even with \( n \geq 2 \). Then the numerator is

\[
(c^2 + c - 3g(c) - 1)H_n^2 + (g(c))^2 + H_n(H_n+1 + H_{n-1} - 1)
\]

\[
\geq H_n(H_n-1 - H_{n-2}) + (g(c))^2 - 1 \geq 0
\]

(the equalities hold only for \( n = 2 \) and \( c = 1 \)). Suppose that \( n \) is odd with \( n \geq 1 \).

Then the numerator is

\[
(3g(c) - c^2 - c + 1)H_n^2 + (g(c))^2 + H_n(H_n+1 + H_{n-1} - 1)
\]

\[
\geq H_n^2 + H_n(H_n+1 + H_{n-1}) + (g(c))^2 - 1 > 0.
\]

Therefore, for all \( n \geq 1 \)

\[
\frac{1}{H_{n-1}H_n + (-1)^n g(c) - 1} > \sum_{k=n}^{\infty} \frac{1}{H_k^2}.
\]

(17)

Similarly,

\[
\frac{1}{H_{n-1}H_n + (-1)^n g(c) + 1} - \frac{1}{H_n^2} - \frac{1}{H_nH_{n+1} + (-1)^{n+1} g(c) + 1} = \frac{H_n^2 + (-1)^{n+1} 2g(c)}{(H_{n-1}H_n + (-1)^n g(c) - 1)(H_nH_{n+1} + (-1)^{n+1} g(c) - 1)} - \frac{1}{H_n^2} \\
= \frac{(-1)^n (c^2 + c - 1 - 3g(c))H_n^2 + (g(c))^2 - H_n(H_n+1 + H_{n-1}) - 1}{H_n^2(H_{n-1}H_n + (-1)^n g(c) - 1)(H_nH_{n+1} + (-1)^{n+1} g(c) + 1)}.
\]

If \( n \) is even, then the numerator is less than or equal to

\[-H_n(H_n+1 + H_n + H_{n-1}) + (g(c))^2 - 1.\]

If \( n \) is odd, then the numerator is less than or equal to

\[-H_n(H_{n-1} - H_{n-2}) + (g(c))^2 - 1.\]
Thus, in any case, for \( n \geq n_5 \) \( (n_5 \text{ is large}) \) both values are negative. Therefore,

\[
\frac{1}{H_{n-1}H_n + (-1)^n g(c) + 1} < \sum_{k=n}^{\infty} \frac{1}{H_k^2}.
\]  

(18)

By Lemma 10, parts (1) and (4)

\[
\frac{1}{H_{n-1}H_n + (-1)^n g(c)} - \frac{1}{H_n^2} = \frac{1}{H_{n+1}^2} - \frac{1}{H_{n+1}H_{n+2} + (-1)^n g(c)}
\]

\[
= \frac{H_{n+1}H_{n+2} - H_{n-1}H_n}{(H_{n-1}H_n + (-1)^n g(c))(H_{n+1}H_{n+2} + (-1)^n g(c))} - \frac{H_n^2 + H_{n+1}^2}{H_n^2 H_{n+1}^2}
\]

\[
= \frac{(cH_{2n} + H_{2n+1})((-1)^n(c^2 + c - 1)H_nH_{n+1}}{H_{n-1}H_n + (-1)^n g(c))(H_{n+1}H_{n+2} + (-1)^n g(c))H_n^2 H_{n+1}^2}
\]

\[
+ \frac{(-1)^{n+1} g(c)(H_{n+1}H_{n+2} + H_nH_{n+1}) - (g(c))^2}{(H_{n-1}H_n + (-1)^n g(c))(H_{n+1}H_{n+2} + (-1)^n g(c))H_n^2 H_{n+1}^2}.
\]

Hence, if \( n \) is even with \( n \geq n_6 \) (large), then by

\[
(c^2 + c - 1)H_nH_{n+1} - g(c)(H_{n+1}H_{n+2} + H_nH_{n+1}) - (g(c))^2 < 0
\]

we have

\[
\frac{1}{H_{n-1}H_n + g(c)} < \sum_{k=n}^{\infty} \frac{1}{H_k^2}.
\]  

(19)

If \( n \) is odd with \( n \geq n_7 \) (large), then by

\[
-(c^2 + c - 1)H_nH_{n+1} + g(c)(H_{n+1}H_{n+2} + H_nH_{n+1}) - (g(c))^2 > 0
\]

we have

\[
\frac{1}{H_{n-1}H_n - g(c)} > \sum_{k=n}^{\infty} \frac{1}{H_k^2}.
\]  

(20)

In conclusion, if \( n \) is even, by (17) and (19) we obtain

\[
\frac{1}{H_{n-1}H_n + g(c)} < \sum_{k=n}^{\infty} \frac{1}{H_k^2} < \frac{1}{H_{n-1}H_n + g(c) - 1}.
\]

If \( n \) is odd, by (18) and (20) we obtain

\[
\frac{1}{H_{n-1}H_n - g(c) + 1} < \sum_{k=n}^{\infty} \frac{1}{H_k^2} < \frac{1}{H_{n-1}H_n - g(c)}.
\]
4. The Sum of Reciprocal Jacobsthal Numbers

It would be interesting to find similar results for the sum \( \sum_{k=n}^{\infty} U_k^{-1} \), where the sequence \( \{U_n\}_n \) is defined by \( U_n = aU_{n-1} + bU_{n-2} \) \( (n \geq 2) \) with \( U_0 = c \) and \( U_1 = d \) for arbitrary fixed integers \( a, b, c \) and \( d \).

Here, we mention the result for the sum of reciprocal Jacobsthal numbers, defined by \( J_n = J_{n-1} + 2J_{n-2} \) \( (n \geq 2) \) with \( J_0 = 0 \) and \( J_1 = 1 \) (Cf. [7, Ch.39]).

**Theorem 11.** We have

\[
\left( \sum_{k=n}^{\infty} \frac{1}{J_k} \right)^{-1} = \begin{cases} 
J_{n-1} - 1 & \text{if } n \text{ is even and } n \geq 2; \\
J_{n-1} & \text{if } n \text{ is odd and } n \geq 1.
\end{cases}
\]

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