LINEAR EQUATIONS INVOLVING ITERATES OF $\sigma(N)$

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Abstract
We study integers $N$ satisfying the equation $\sigma(\sigma(N)) = A\sigma(N) + BN$.

1. Introduction
We denote by $\sigma(N)$ the sum of divisors of $N$. $N$ is said to be perfect if $\sigma(N) = 2N$ and multiperfect if $\sigma(N) = kN$ for some integer $k$. It is not known whether or not an odd perfect/multiperfect number exists. There are many known conditions which must be satisfied by such a number. But these results are far from answering whether or not an odd perfect/multiperfect number exists.

Suryanarayana [7] called $N$ superperfect if $\sigma(\sigma(N)) = 2N$ and Pomerance [5] called $N$ super multiply perfect if $\sigma(\sigma(N)) = kN$ for some integer $k$. More generally, Cohen and te Riele [2] defined $N$ to be $(m,k)$-perfect if $\sigma^{(m)}(N) = kN$, where $\sigma^{(m)}(N)$ denotes $\sigma(\sigma^{(m-1)}(N))$ with $\sigma^{(0)}(N) = N$.

We introduce another analogous notion of perfect/multiperfect numbers. We say that $N$ is $(n; a_0 : \ldots : a_n)$-perfect if $\sum_{i=0}^{n} a_{n-i} \sigma^{(i)}(N) = 0$ and $(n; a_1, \ldots, a_n)$-perfect if $N$ is $(n; -1 : a_1 : \ldots : a_n)$-perfect. In particular, $N$ is $(2; A, B)$-perfect if $\sigma(\sigma(N)) = A\sigma(N) + BN$.

We begin by noting that almost all integers are $(2; A, B)$-perfect for some integers $A, B$. This fact follows from Katai and Subbarao [3, Theorem 1]. They proved that for any fixed $m \geq 1$, we have $(N, \sigma^{(m)}(N)) = \prod_{p || N} p^k$, where $p$ runs over all primes below $x_m^2$, for all integers $N \leq x$ with $o(x)$ exceptions (Here we use the notation $x_0 = x, x_{i+1} = \max\{1, \log x_i\}$ for all $i \geq 0$, where $\log x$ is the natural logarithm of $x$, introduced in [4]). Hence, for almost all integers $N$, $(N, \sigma(N))$ divides $\sigma^{(2)}(N)$, which implies that there exist some integers $A, B$ such that $\sigma(\sigma(N)) = A\sigma(N) + BN$.

On the other hand, we can show that for any fixed positive integers $A, B$, the set of $(2; A, B)$-perfect numbers has density zero.

Theorem 1.1. Let $A, B, C$ be integers not all zero and satisfying $AC \leq 0$. Then the number of $(2; A : B : C)$-perfect numbers below $x$ is at most $x \exp(-(2^{-1/3} + o(1))(\log x)^{1/3}(\log \log x)^{2/3})$.

Our argument is similar to the argument of Pomerance [6] to study the distribution of integers $n$ satisfying $\sigma(n) \equiv a \pmod{n}$. His argument rests on the fact that
almost all integers \( n \) can be written as \( mp \), where \( p \nmid m \) is prime, large and uniquely determined by \( m \) except some special cases. Our argument adopts a factorization of \( p + 1 \) into \( lq \) with \( q \) large and \( (l, q) = 1 \), to show that \( q \) is uniquely determined by \( m \) and \( l \) under some condition.

We note that this result does not seem to apply to the case \( AC > 0 \). If \( p, 2p - 1 \) are both prime, then \( n = 2p - 1 \) satisfies \( 2\sigma(\sigma(n)) - 3\sigma(n) + 6n = 0 \) and therefore \( n \) is \( (2; 2 : -3 : 6) \)-perfect. More generally, we can easily confirm the following result.

**Theorem 1.2.** Let \( R \) be an arbitrary positive integer. If \( p \) and \( N = Rp - 1 \) are both prime, then \( N \) is \( (2; R : -(R + 1)\sigma(R) : R(\sigma(R))) \)-perfect.

**Corollary 1.3.** If \( R \) is a \( k \)-multiperfect number and \( p, N = Rp - 1 \) are both prime, then \( N \) is \( (2; k(R + 1) : -kR) \)-perfect.

A well-known conjecture states that, for any even integer \( R \), the number of primes \( p < x \) with \( Rp - 1 \) also prime is asymptotically equal to \( cx/(\log x)^2 \) for some constant \( c > 0 \) depending on \( R \), contrary to the above given estimate \( O(x \exp(-\frac{2^{1/3} + o(1)}{(\log x)^{1/3}}(\log \log x)^{2/3})) \).

### 2. Notations and Preliminary Lemmas

We denote by \( P(n), p(n) \) the largest and smallest prime factor of \( n \) respectively. For the positive real number \( x \), let us denote \( x_0 = x, x_{i+1} = \max\{1, \log x_i\} \) as mentioned in the previous section. We denote by \( c \) some positive constant not necessarily same at every occurrence. Furthermore, we denote by \( x, y, z \) real numbers and we put \( u = \log x/\log y \) and \( v = \log y/\log z \).

**Lemma 2.1.** Denote by \( \Psi(x, y) \) the number of integers \( n \leq x \) divisible by no prime \( > y \). If \( y > x^{2/3} \), then we have \( \Psi(x, y) < x \exp(-(1 + o(1))u \log u) \) as \( x, u \) tend to infinity.

**Proof.** This follows from a well-known theorem of de Bruijn [1]. For details on the distribution of integers free from large prime factors, we refer the readers to [8, Chapter III. 5], where a simple proof of the lemma is also given. \( \square \)

**Lemma 2.2.** Let

\[
s(x,k) = \sum_{p \leq x, p \equiv -1 \pmod k} \frac{1}{p},
\]

Uniformly in \( k \) and \( x \geq e^2 \), we have

\[
s(x, k) \ll \frac{x^2}{\varphi(k)}.
\]
Proof. This inequality can be immediately obtained using partial summation and the Brun–Titchmarsh inequality. The complete proof is given in [4, Lemma 2]. □

Lemma 2.3. Let \( S(x) = \{n \mid n \leq x, p^n \mid n \mbox{ for some } p, a \mbox{ with } p^a > y, a \geq 2 \} \). Then we have \( \#S(x) \ll x y^{-1/2} \).

Proof. Let \( \Pi(t) \) be the number of perfect powers below \( t \). It is clear that \( \Pi(t) < t^{1/2} + t^{1/3} + \cdots + t^{1/k} < t^{1/2} + ct^{1/3} \log t \ll t^{1/2} \), where \( k = \lfloor (\log x)/(\log 2) \rfloor \).

Let us denote by \( \gamma_p \) the smallest integer \( \gamma \) for which \( p^\gamma > y \) and \( \gamma > 1 \). Clearly we have \( \#S(x) \leq x \sum_{p \leq x} p^{-\gamma_p} \). Since \( p^\gamma > y \), we have by partial summation

\[
\sum_{p \leq x} \frac{1}{p^\gamma} \leq \frac{\Pi(x)}{x} - \frac{\Pi(y)}{y} + \int_y^x \frac{\Pi(t)}{t^2} dt \ll x^{-1/2}.
\]

This proves the lemma. □

Lemma 2.4. If \( y > z > 2x^2 \) and \( v > x_2 \), then the number of integers \( \leq x \) divisible by some prime \( p \geq y \) with \( P(p+1) < z \) is at most \( x \exp(-(1+o(1))v \log v) \).

Proof. The number of such integers is at most

\[
\sum_{y \leq p \leq x, P(p+1) < z} \frac{x}{p} \leq x \sum_{y \leq m \leq x, P(m+1) < z} \frac{1}{m},
\]

where \( p \) and \( m \) respectively run over primes and integers satisfying the described conditions. By partial summation, we find that the last sum is at most

\[
\frac{\Psi(y, z)}{y} + \int_y^{2x} \frac{\Psi(t, z)dt}{t^2}.
\]

Since we have \( \Psi(t, z)/t < \exp(-(1+o(1))v \log v) \) uniformly for \( t \in [y, 2x] \) by Lemma 2.1, the last sum in (4) can be bounded from above by

\[
\exp(-(1+o(1))v \log v) \left(1 + \int_y^{2x} \frac{dt}{t}\right).
\]

This integral is \( O(x_1) = O(\exp v) \) since, by assumption, \( x_2 < v \). Thus we obtain

\[
x \sum_{y < m \leq x, P(m+1) < z} \frac{1}{m} = x \exp(-(1+o(1))v \log v).
\]

This completes the proof. □
3. Proof of Theorem 1.1

Let $y, z$ be real numbers and put $u = \log x / \log y$ and $v = \log y / \log z$. We choose $y, z$ later so that $x > y > z > x_1^2, v > x_2$ and $u, v, y, z$ tend to infinity as $x$ does so.

Let

$$S_1 = \{ n \mid n \leq x, P(n) \leq y \}$$

and

$$S_2 = \{ n \mid n \leq x, p^a \mid n \text{ for some } p, a \text{ with } p^a \geq z, a \geq 2 \}. \quad (9)$$

We immediately obtain $\#S_1 = O(x \exp(-(1 + o(1))u \log u))$ by Lemma 2.1 and $\#S_2 = O(x/z^{1/2})$ by Lemma 2.3.

Denote by $S_3$ the set of integers $n \leq x$ not in $S_1 \cup S_2$ which can be written in the form $mp$, where $p$ is a prime $\geq y$, $m$ is an integer not divisible by $p$ and $(\sigma(m), p+1)$ is divisible by some prime $q \geq z$.

Let $n$ be an integer in $S_3$ and write $n = mp$ in the above way. Then $q \mid \sigma(r^a)$ for some prime $r$ dividing $m$ and some integer $a$ with $r^a \mid m$. Since $q \geq z$, we have $a = 1$ by the assumption $n \notin S_2$. Hence, $m$ is divisible by some prime $r$ congruent to $-1 \pmod{q}$.

Since $m \leq x/p$, the number of integers $n$ satisfying $n = mp, q \mid (\sigma(m), p+1)$ for some $q \geq z$ is at most

$$\sum_{q \geq z \mid p \equiv 1 \pmod{q}} \sum_{r \equiv -1 \pmod{q}} \sum_{x/p \mid (q) \mid r} x/p \leq \sum_{q \geq z} \frac{c x x_2}{q^2} \leq \frac{c x x_2}{z \ln z}, \quad (10)$$

by Lemma 2.2. Hence, we have $\#S_3 = O(x x_2^2/z)$.

Denote by $S_4$ the set of integers $n \leq x$ divisible by some prime $p \geq y$ with $P(p+1) < z$ or $q^2 \mid (p+1)$ for some $q \geq z$. By Lemma 2.4, the number of integers $n \leq x$ divisible by some prime $p \geq y$ with $P(p+1) < z$ is $O(x \exp(-(1 + o(1))v \log v))$. The number of integers $n \leq x$ divisible by some prime $p$ with $q^2 \mid (p+1)$ for some $q \geq z$ is at most

$$x \sum_{q \geq z \mid p \equiv 1 \pmod{q}} \sum_{(mod q^2)} \frac{1}{p} \leq c x x_2 \sum_{q \geq z} \frac{1}{q} \leq \frac{c x x_2}{z x_1^{1/3}} \ll \frac{x}{z}, \quad (11)$$

by the assumption that $z > x_1^2$.

Combining these estimates yield $\#S_4 = O(x(1/z + \exp(-(1 + o(1))v \log v)))$.

We may assume that at least one of $A$ and $C$ is nonzero since the equation

$$A \sigma(\sigma(N)) + B \sigma(N) + CN = 0$$

does not hold if exactly one of $A, B, C$ is nonzero.

Now let us denote by $T$ the set of $(2; A : B : C)$-perfect numbers $n \leq x$ belonging to none of $S_i(i = 1, 2, 3, 4)$. We assume that $n \in T$. Since $n \notin S_1 \cup S_2$, we have $P(n) > y$ and $P(n)^2 \mid n$. Thus $n$ can be expressed as $n = mp, p > y$ and $p \nmid m$.

Now it follows from $n \notin S_3$ that $(\sigma(m), p+1)$ has no prime factor $\geq z$. Let $T_m$ denote the set of such integers. Moreover, we write $p + 1 = N_1 N_2$ in the way
$P(N_1) < z \leq p(N_2)$ and divide each of $T_m$ into sets $T_{m,N_1}$ according to the value of $N_1$. Since $n \notin S_4$, we have $p + 1$ is divisible by some prime $Q \geq z$ exactly once. By the definition of $N_1, N_2$, we have $N_2 = N_3 Q$ and $T_{m,N_1}$ can be covered by sets $T_{m,N_1,N_3}$ according to the value of $N_3$.

We shall show each $T_{m,N_1,N_3}$ consists at most one element. We have $\sigma(n) = \sigma(m)(p + 1)$ and $\sigma(\sigma(n)) = \sigma(M_1N_1)\sigma(M_2)\sigma(N_2)$, where $M_1, M_2$ are uniquely determined by $\sigma(m) = M_1M_2$, $P(M_1) < z \leq p(M_2)$. Furthermore, noting that $Q$ does not divide $(p + 1)/Q$ by the assumption that $n \notin S_4$, we obtain $\sigma(N_2) = \sigma(N_3)(Q + 1)$. Since $A\sigma(\sigma(n)) + B\sigma(m)(p + 1) + Cmp = 0$, we have

$$A\sigma(M_1N_1)\sigma(M_2)\sigma(N_3)(Q + 1) + (B\sigma(m) + Cm)N_1N_3Q - Cm = 0.\quad (12)$$

Denoting

$$C_1 = A\sigma(M_1N_1)\sigma(M_2)\sigma(N_3),\quad C_2 = (B\sigma(m) + Cm)N_1N_3,\quad C_3 = Cm,$\quad (13)$$

this equation can be written

$$(C_1 + C_2)Q = (C_3 - C_1).\quad (14)$$

Since $AC \leq 0$, we have $C_1 \neq C_3$ and therefore $Q$ can be uniquely determined as

$$Q = \frac{C_3 - C_1}{C_1 + C_2}.\quad (15)$$

This is the crucial point where we use the assumption $AC \leq 0$. The uniqueness of $Q$ yields that $\#T_{m,N_1,N_3} \leq 1$ for any $m, N_1, N_3$.

By the definition of $T_{m,N_1,N_3}$, each element of $T$ must belong to $T_{m,N_1,N_3}$ for at least one triple $(m, N_1, N_3)$. Since $N_1N_3 = (p + 1)/Q \leq (x/m + 1)/z \leq 2x/(mz)$, we have

$$\#T \leq \sum_{m \leq x/y} \sum_{N_1 \leq x/y} \sum_{N_3 \leq z^2/mz} \frac{2x}{mz} \leq \frac{cxz}{z^2/mz} \leq \frac{cxz}{z^{3/2}}.\quad (16)$$

Now we conclude that the number of $(2; A, B)$-perfect numbers $\leq x$ is at most $\#S_1 + \#S_2 + \#S_3 + \#S_4 + \#T$, which is

$$O(x(z^{-1/2} + \exp(-(1 + o(1))u \log u) + \exp(-(1 + o(1))v \log v)))\quad (17)$$

by those estimates for $\#S_i$’s and $\#T$ given above.

In order to search the optimal estimate, we put $\log z = c_1x_1^{1/3}x_2^{2/3}, \log y = c_2x_1^{2/3}x_2^{1/3}$ and we have $u \log u = (1 + o(1))c_1^{-1}x_1^{1/3}x_2^{2/3}$ and $v \log v = (1 + o(1))(c_2/c_1)x_1^{1/3}x_2^{2/3}$. We see that $\max\{c_1/2, c_2/c_1, 1/c_2\} \geq 2^{-1/3}$ with the equality attained when we choose $c_1 = 2^{2/3}, c_2 = 2^{1/3}$. This choice gives the desired estimate. This completes the proof of Theorem 1.1.

We remark that $(2; R : -(R+1)\sigma(R) : R\sigma(R))$-perfect numbers given in Theorem 1.2 correspond to the case $m = N_1 = 1, C = A\sigma(R) = -(B + C)R$. 

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References


[6] C. Pomerance, *On the congruences \(\sigma(n) \equiv a \pmod{n}\) and \(n \equiv a \pmod{\varphi(n)}\)*, Acta Arith. **26** (1975), 265–272.
