THE FINITE HEINE TRANSFORMATION

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Abstract
We shall present finite summations that converge to the Heine $2\phi_1$ transformations in the limit as $n \to \infty$. We shall investigate their partition-theoretic implications.

1. Introduction

In an expository article describing Euler’s pioneering work on partitions, I was particularly drawn to Euler’s assertion [6, p. 566, eq. (5.2) corrected]

$$\prod_{n=0}^{\infty} \left(q^{-3^n} + 1 + q^{3^n}\right) = \sum_{n=-\infty}^{\infty} q^n,$$  \hfill (1.1)

an identity valid only in a formal sense in that neither the series nor the product converges for any value of $q$.

This led to my comparisons of the two infinite series identities ([6, p. 567, eq. (5.5)] and [6, p. 567, eq. (5.6)] respectively):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$  \hfill (1.2)

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)/2}. \hfill (1.3)$$

Each of the left-hand series is analytic inside $|q| < 1$ with $|q| = 1$ as a natural boundary, and the second series is formally transformable into the first by the mapping $q \to 1/q$. The fact that $|q| = 1$ is a natural boundary means we should not be surprised when the same transformation applied to the right-hand side produces only nonsense.

However, it was observed in [4] that it is sometimes possible to find polynomial or rational function identities that converge to infinite $q$-series in the limit. This

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observation in [7] was the secret to dealing with Regime II of Baxter’s generalized hard-hexagon model (cf. [5, Ch. 8]).

So this led to the question: Are there finite identities that would both (A) simplify to (1.2) and (1.3) in the limit, and (B) allow the mapping \( q \to 1/q \) prior to taking limits?

The answer to this question is yes. In Section 2 we provide \( q \)-analogs of the Heine transformations of the \( _2\phi_1 \). In Section 3, we shall derive generalizations of the following corollaries.

\[
\sum_{n=0}^{N} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} = \prod_{n=1}^{N} \frac{1}{(1-q^n)} \sum_{j=0}^{N} \frac{q(N+1)^j}{(1-q)(1-q^2) \cdots (1-q^l)}, \quad (1.4)
\]

and

\[
\sum_{n=0}^{N} \frac{q^n}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} = \prod_{n=1}^{N} \frac{1}{(1-q^n)} \sum_{j=0}^{N} \frac{(-1)^jq^{(j+1)/2}}{(1-q)(1-q^2) \cdots (1-q^{N+j})}. \quad (1.5)
\]

Clearly (1.4) and (1.5) converge to (1.2) and (1.3) as \( N \to \infty \), and by reversing the sum on the right-hand side it is a simple matter to see that (1.4) becomes (1.5) under the now legitimate mapping \( q \to 1/q \).

In Section 4, we shall note quite transparent combinatorial proofs of (1.4) and (1.5).

2. Finite Heine Transformations

We shall employ the following standard notation

\[
(a)_n = (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (2.1)
\]

\[
(a_1, \ldots, a_r; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_r; q)_n, \quad (2.2)
\]

and

\[
_{r+1}\phi_r \left( \begin{array}{c} a_0, a_1, \ldots, a_r; q, t \\ b_1, \ldots, b_r \end{array} \right) = \sum_{n=0}^{\infty} \frac{(a_0, a_1, \ldots, a_r; q)_n t^n}{(q, b_1, \ldots, b_r; q)_n}. \quad (2.3)
\]

Lemma 1. For non-negative integers \( n \),

\[
_{3}\phi_2 \left( \begin{array}{c} q^{-n}, \alpha, \beta; q, q \\ \gamma, q^{1-n}/\tau \end{array} \right) = \frac{(\alpha\tau; q)_n}{(\tau; q)_n} _{3}\phi_2 \left( \begin{array}{c} q^{-n}, \gamma/\beta, \alpha; q, \beta \tau q^n \\ \gamma, \alpha \tau \end{array} \right). \quad (2.4)
\]

Proof. In (III.13) of [8, p. 242], set \( b = \gamma/\beta \), \( c = \alpha \), \( d = \gamma \), \( e = \alpha \tau \). The result after simplification is (2.4). \( \square \)
Theorem 2. For non-negative integers \( n \),
\[
3\phi_2\left(\frac{q^{-n}, \alpha; q}{\gamma, q^{1-n}/\tau}\right) = \frac{(\beta, \alpha\tau; q)_{\gamma, q^{1-n}/\tau}}{(\gamma, \tau; q)_{\gamma, q^{1-n}/\tau}} 3\phi_2\left(\frac{q^{-n}, \gamma/\beta, \tau; q, q}{\alpha\tau, q^{1-n}/\beta}\right). \tag{2.5}
\]

Remark. When \( n \to \infty \), this is Heine’s classic \( 2\phi_1 \) transformation [8, p. 9, eq. (1.4.1)], [3, p. 28, Cor. 2.3].

Proof. If in Lemma 1, we replace \( \alpha, \beta, \gamma, \) and \( \tau \) by \( \gamma/\beta, \tau, \alpha\tau \) and \( \beta \) respectively, we find that
\[
3\phi_2\left(\frac{q^{-n}, \gamma/\beta, \alpha; q, \beta\tau q}{\gamma, \alpha\tau}\right) = \frac{(\beta; q)_{\gamma, \alpha\tau}}{(\gamma, \tau; q)_{\gamma, \alpha\tau}} 3\phi_2\left(\frac{q^{-n}, \gamma/\beta, \tau; q, q}{\alpha\tau, q^{1-n}/\beta}\right). \tag{2.6}
\]
Now substituting the left-hand side of (2.6) into the right-hand side of (2.4) we deduce (2.5). \( \square \)

Corollary 3. For non-negative integers \( n \),
\[
3\phi_2\left(\frac{q^{-n}, \alpha; q}{\gamma, q^{1-n}/\tau}\right) = \frac{(\gamma/\beta, \alpha\tau; q)_{\gamma, q^{1-n}/\gamma}}{(\gamma, \tau; q)_{\gamma, q^{1-n}/\gamma}} 3\phi_2\left(\frac{q^{-n}, \alpha\beta/\gamma, \beta; q, q}{\beta\tau, \beta q^{1-n}/\gamma}\right). \tag{2.7}
\]
Proof. Apply Theorem 2 (with \( \alpha, \beta, \gamma \) and \( \tau \) replaced by \( \gamma/\beta, \alpha\tau \) and \( \beta \) respectively) to transform the \( 3\phi_2 \) on the right-hand side of (2.5). \( \square \)

Corollary 4. For non-negative integers \( n \),
\[
3\phi_2\left(\frac{q^{-n}, \alpha; q}{\gamma, q^{1-n}/\tau}\right) = \frac{(\frac{\alpha\beta\tau}{\gamma}; q)_{\tau, q}}{(\tau; q)_{\tau, q}} 3\phi_2\left(\frac{q^{-n}, \frac{\alpha}{\beta}, \frac{2}{\beta}; q, q}{\gamma, q^{1-n}/(\alpha\beta\tau)}\right). \tag{2.8}
\]
Proof. Apply Theorem 2 (with \( \alpha, \beta, \gamma \) and \( \tau \) replaced by \( \beta, \alpha\beta/\gamma, \beta\tau, \gamma/\beta \) respectively) to transform the \( 3\phi_2 \) on the right-hand side of (2.7). \( \square \)

Corollaries 3 and 4 reduce to the second and third Heine transformations [8, p. 10] when \( n \to \infty \).

3. Identities (1.4) and (1.5)

Theorem 5. For non-negative integers \( n \),
\[
\sum_{j=0}^{n} \frac{q^j}{(q, \gamma; q)_j} = \frac{1}{(\gamma)_n} \sum_{j=0}^{n} \frac{(-1)^j q^{j(j-1)/2}}{(q)_{n-j}}. \tag{3.1}
\]
Proof. Set $\alpha = 0$, $\tau = q$ and let $\beta \to 0$ in Theorem 2. The desired result follows after algebraic simplification.

Theorem 6. For non-negative integers $n$,

$$
\sum_{j=0}^{n} \frac{q^{j^2} \gamma^j}{(q, q; q)_j} = \frac{1}{(\gamma q)_n} \sum_{j=0}^{n} \frac{\gamma^j q^j (n+1)}{(q)_j}.
$$

(3.2)

Proof. Replace $q$ by $1/q$ and $\gamma$ by $1/q\gamma$ in (3.1), then reverse the sum on the right-hand side and simplify.

Identity (1.5) is Theorem 5 with $\gamma = q$, and (1.4) is Theorem 6 with $\gamma = 1$.

4. Combinatorial Proofs

Replacing $q$ by $q^2$ in Theorem 5 and then replacing $\gamma$ with $-\gamma q$, we see that Theorem 5 is equivalent to the following assertion:

$$
\sum_{j=0}^{n} \frac{q^{2j} (-\gamma q^{2j+1}; q^2)_{n-j}}{(q^2, q^2)_j} = \sum_{j=0}^{n} \frac{\gamma^j q^{j^2}}{(q^2; q^2)_{n-j}}.
$$

(4.1)

Proof of (4.1). The left-hand side of (4.1) is the generating function for partitions in which (1) all parts are $\leq 2n$, (2) odd parts are distinct, and (3) each odd is $> j$ each even. The general two-modular Ferrers graph [3, p. 13] for such partitions is thus

$$
\begin{array}{cccccccccccc}
2 & 2 & \cdots & 2 & \cdots & 2 & \cdots & 2 & 1 \\
2 & 2 & \cdots & 2 & \cdots & \cdots & 2 & 1 \\
2 & 2 & \cdots & 2 & \cdots & \cdots & 2 & 1 \\
& & & \cdots & \cdots & \cdots & 2 & 1 \\
2 & 2 & \cdots & 2 & 2 \\
2 & 2 & \cdots & 2 \\
\vdots \\
2
\end{array}
$$

Now remove the columns that have a 1 at the bottom. In light of the fact that the odds were distinct, we see that if there were originally $j$ odd parts, then we have removed $1 + 3 + 5 + \cdots + (2j - 1) (= j^2)$. The remaining parts are all even and the largest is at most $2n - 2j$. Thus this transformation (which is clearly reversible) provides the partitions generated by the right-hand side of (4.1) and thus we have a bijective proof of Theorem 5.

Proof of (3.2). Classical arguments immediately reveal that the left-hand side of (3.2) is the generating function for partitions with Durfee square of side at most $n$. $\gamma$ keeps track of the number of parts.
On the other hand, the side of the Durfee square is the largest \( j \) such that the \( j^{th} \) part is \( \geq j \). So we may replicate the partitions generated by the left-hand side of (3.2) by exhibiting the generating function for partitions in which the parts \( > n \) are at most \( n \) in number. If there are \( j \) parts greater than \( n \), the generating function is

\[
\frac{\gamma^j q^{j(n+1)}}{(\gamma q)_n(q)_j}.
\]

Hence summing on \( j \) from 0 to \( n \) we obtain a new expression for the generating function for partitions with Durfee square at most \( n \), and this proves (3.2).

5. Conclusion

There are many other corollaries obtainable from the finite Heine transformations. The \( q \)-Pfaff-Saalschütz summation is merely [8, p. 13, eq. (1.7.2)] with \( \tau = \gamma/\alpha\beta \). One can also obtain a finite version of the \( q \)-analog of Kummer’s theorem [2], however, the result does not reduce to the hoped for “sum equals product” identity. Also it should be possible to provide a fully combinatorial proof of Theorem 2 along the lines given in [1] for the \( n \to \infty \) case.

References


