Additional Topics

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Here, I describe some of the specific topics that I am interested in and, working on or intend to work on in the near future. In general, I am interested in problems from a variety of topics such as extremal graph theory, set systems, and combinatorial geometry, among others.

Graph coloring extensions As mentioned earlier, graph coloring is highly applicable, and at times these applications have motivated some extensions of the notion of graph coloring. I have already worked on $L(2,1)$-labeling. Another such interesting problem is strong edge-coloring, an edge coloring in which edges at distance 1 or 2 receive distinct colors. In 1985, Erdős and Nesetril conjectured that $s'(G)$, smallest number of colors needed in such a coloring, is at most $(5/4)\Delta^2(G)$. Faudree, et al. [5] conjectured that $s'(G) \leq \Delta^2(G)$ when G is bipartite. Some partial progress has been made on these and other such conjectures. I intend to work on this topic and, as a start I showed that $s'(K(2k+1,k)) = 2k+1$ in [14].

A related generalization is $\chi(G^2)$, where G^2 is the graph on $V(G)$ with two vertices adjacent if they are at distance 1 or 2 in G. Strong edge-coloring can be thought of as the study of $\chi(G^2)$ for line graphs. It is also related to $L(2,1)$-labeling. Since $\Delta^2(G)$ is also an upper bound on $\Delta(G^2)$, I expect $\chi(G^2)$ to be a good lower bound on $\lambda(G)$. This makes the study of $\chi(G^2)$ worthwhile. For instance, finding a lower bound on $\chi(K(2k+1,k)^2)$ or its easier variation, an upper bound on the independence number of $K(2k+1,k)^2$, are both challenging questions that could lead to non-trivial lower bounds on $\lambda(K(2k+1,k))$ complementing the upper bound mentioned earlier.

Euclidean Ramsey Theory. Ramsey theory typically deals with problems of the following type. Given a set S, a family $\mathcal{F}$ of subsets of S, and $r \in \mathbb{Z}^+$, is it true that in every partition of $S = C_1 \cup \cdots \cup C_r$ into r subsets, these is some C_i that contains some $F \in \mathcal{F}$. In Euclidean Ramsey theory, S is taken to be some Euclidean space, and the sets in $\mathcal{F}$ are determined by various geometric configurations. This field was started by a series of papers by Erdős, Graham, Montgomery, Spencer, etc. (see [11] for a survey).

Recalling its definition, metric space coloring turns out to be a special type of Euclidean Ramsey problem. Due to this connection, I have become interested in this area. This field provides a common framework for a variety of topics from geometry and number theory, and it has a lot of interesting open problems that I intend to work on.

The Kneser graph $K(m,k)$. As mentioned in the section on $L(2,1)$-labeling of graphs, the Kneser graph is an interesting graph to explore for its own sake, especially for its relationship to many topics. For instance, it can be found in connection with combinatorial geometry in the papers of Lovasz [17] and Bárány [2], in connection with extremal combinatorics in Füredi–Griggs–Kleitman [7], intersecting families in Erdős-Ko-Rado [4], Frankl-Füredi [6], etc. In my research, I have studied its $L(2,1)$-labeling, strong edge-coloring, and security number, and I also believe that it has relevance to finding good lower bounds on $\chi(\mathbb{Z}_1^n, r)$. I am also interested in the conjecture that the distinguishing number (least number of labels needed to break all symmetries) of Kneser graph $K(m,k)$ is 2, for m large enough. So, I would like to further study problems about the Kneser graph and explore its connections with other problems.

Intersecting families. Bounding the size of a family whose members satisfy certain intersection properties is one of the most fundamental problems in combinatorics of finite sets. Such results are widely applicable and provide good extremal constructions (see Babai and Frankl [1] for numerous such results). We used the Frankl-Wilson result on intersecting families to construct a lower bound on $\chi(\mathbb{R}_p^n, 1)$. A better or different such result might be needed to improve the lower bounds on $\chi(\mathbb{R}_p^n, 1)$. Also, for $L(2,1)$-labeling, a lower bound on $\chi(K(2k+1,k)^2)$ could be reformulated as "what is the maximum size of a k-uniform family $\mathcal{F}$ of subsets of a $2k+1$-element set, satisfying the condition $1 \leq |F \cap F'| \leq k-1$ for all $F, F' \in \mathcal{F}$?". Such an $\mathcal{F}$ is an independent set in $(K(2k+1,k))^2$.

Packing and covering of triangles. Let $\nu(G)$ be the maximum size of a set of pairwise edge-disjoint triangles of G, and let $\tau(G)$ be the minimum number of edges that cover all the triangles of G. We want to bound $\tau(G)$ in terms of $\nu(G)$. Note that $\tau(G)$ has trivial lower bound $\nu(G)$ and upper bound $3\nu(G)$. The graphs $G = K_4$ and K_5 show that $\tau(G)$ can be as large as $2\nu(G)$. In 1981, Zs. Tuza [20] conjectured that $\tau(G) \leq 2\nu(G)$, which has drawn a great deal of attention. Despite some progress, the conjecture still remains open. I am interested in such packing and covering problems on graphs and hypergraphs.

Transversals of convex bodies. *to be added*

Lottery problem. *to be added*

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