

Combinatorial geometry is the study of combinatorial properties of fundamental geometric objects, whose origins go back to antiquity. It has come into maturity in the last century through the seminal works of O. Helly, K. Borsuk, P. Erdős, H. Hadwiger, L. Fejes Tóth, B. Grünbaum and many other excellent mathematicians who initiated new combinatorial approaches to classical questions studied by Newton, Gauss, Minkowski, and Hilbert, as well as new areas of investigation. The textbooks by Matousek [12] and, Pach and Agarwal [15] provide an overview of the topics and methods.

1 Coloring metric spaces

Combinatorial geometry is the study of combinatorial properties of fundamental geometric objects, whose origins go back to antiquity. It has come into maturity in the last century through the seminal works of O. Helly, K. Borsuk, P. Erdős, H. Hadwiger, L. Fejes Tóth, B. Grünbaum and many other excellent mathematicians who initiated new combinatorial approaches to classical questions studied by Newton, Gauss, Minkowski, and Hilbert, as well as new areas of investigation. The textbooks by Matousek [12] and, Pach and Agarwal [15] provide an overview of the topics and methods.

Background. A fundamental approach to studying the combinatorial structure of a metric space is : If we partition a space into a small number of parts (i.e., color its points), at least one of these parts must contain certain “unavoidable” point configurations. In the most basic case, the configuration consists of a pair of points at a given distance (which can usually be scaled to one). The first such question, known as the Hadwiger–Nelson problem, was: What is the minimum number of colors needed for coloring the plane so that no two points at unit distance receive the same color? The answer is known to be between 4 and 7, with no improvement in the last 50 years. The same question can be asked for any metric space. This has turned out to be a challenging problem with very little progress made on it. See Pach [14] for a short survey. We consider ℓ_p -spaces and investigate $\chi(\mathbb{R}_p^n, 1)$, i.e., the chromatic number of the unit distance graph on \mathbb{R}^n under the ℓ_p -norm. Another natural candidate is the discrete space \mathbb{Z}^n . In this case, the ℓ_1 -norm is the most natural metric and, since distances cannot be scaled to one, we consider the graph (\mathbb{Z}_1^n, r) on \mathbb{Z}^n with two points adjacent if their ℓ_1 -distance is a fixed integer r . Almost nothing is known about these chromatic numbers, with the only important results for the Euclidean space ($p = 2$), $(1.2 + o(1))^n \leq \chi(\mathbb{R}_2^n, 1) \leq (3 + o(1))^n$. The lower bound was given by Frankl and Wilson [5], and the upper bound by Larman and Rogers [11] in 1981 and 1972, respectively. Other than this, only weak bounds on $\chi(\mathbb{R}_2^3, 1)$ are known. The subgraph induced by rational points has also been investigated in small dimensions: $\chi(\mathbb{Q}_2^2, 1) = 2$ (Woodall [17]), $\chi(\mathbb{Q}_2^3, 1) = 2$ and $\chi(\mathbb{Q}_2^4, 1) = 4$ (Benda and Perles [2]) are the main results. Morayne [13] used Fermat’s Last Theorem to observe that $\chi(\mathbb{Q}_p^2, 1) = 2$ for $p \geq 3$.

Results and future work. In [6, 7, 9], Füredi and I investigate $\chi(\mathbb{R}_p^n, 1)$ for general $p \geq 1$, by studying certain fundamental geometric ideas, namely, partitions of \mathbb{R}^n , sphere packing in \mathbb{R}^n , and covering of \mathbb{R}^n by translates of a convex body, which respectively led to the upper bounds, $\sqrt{p/(2\pi n)}(5(ep)^{1/p})^n$ (for all n), 9^n (for all n), and $c(n \ln n)5^n$ (for large

n). The third upper bound requires a covering with special characteristics, described in the following section. We use an intersection inequality of Frankl and Wilson [5] to construct a finite subgraph of $(\mathbb{R}_p^n, 1)$ with small independence number and, consequently, large chromatic number, which gives a lower bound of $(1.139)^n$ for all p .

Since the graph (\mathbb{Z}_1^n, r) is bipartite when r is odd, we study $\chi(\mathbb{Z}_1^n, r)$ for even r . In [6], Füredi and I prove that $\chi(\mathbb{Z}_1^n, 2) = 2n$ for all n , but $\chi(\mathbb{Z}_1^n, r) \geq 2n+1$ for $n \geq 3$ and even $r \geq 4$. In addition, we give an exponential lower bound $(1.139)^n$ by constructing an appropriate finite subgraph, again using the Frankl-Wilson inequality. Since (\mathbb{Z}_1^n, r) is a subgraph of (\mathbb{R}_1^n, r) , we have $\chi(\mathbb{Z}_1^n, r) \leq \chi(\mathbb{R}_1^n, r) = \chi(\mathbb{R}_1^n, 1)$, so the upper bounds for $\chi(\mathbb{R}_1^n, 1)$ hold here. For small values of r , we have a more useful upper bound, $3r^{n-2}$ for all n and all even $r \geq 4$, which follows from a recurrence that extends a coloring on a selected hyperplane to the whole space.

The exponential lower bound follows from an application of the Frankl-Wilson Inequality to a carefully defined subset of $\{0, 1\}^n$ under scaling. We try to improve the lower bound, possibly by considering a bigger subspace like $\{-1, 0, 1\}^n$.

To improve the upper bounds, the first step could be to combine the packing and covering arguments used in [9] and [7]. A fundamental improvement might depend on a better understanding of the coloring of the $(n-1)$ -dimensional sphere of radius a as a subgraph of \mathbb{R}^n . Except for a lower bound of n by Lovász (see [14]) and some bounds for small dimensions, almost nothing is known; we plan to further investigate the problem.

2 Covering Euclidean n -space

Roughly speaking, the *density* of a covering of \mathbb{R}^n is the average number of bodies needed to cover a given unit space. The density of a cover measures how good the cover is. We seek to minimize the density of a collection \mathcal{C} of convex bodies that covers \mathbb{R}^n .

Rogers [16] proved that, for a given closed convex body C in the n -dimensional Euclidean space, where $n \geq 3$, there is a covering for \mathbb{R}^n by translates of C with density $O(n \ln n)$. However, low density does not imply low multiplicity of the covering, where *multiplicity* is the number of copies of $C \in \mathcal{C}$ containing each point. Even though the global density of a covering is low, there can exist local clusters of high multiplicity. Later, Erdős and Rogers [4] showed that, for sufficiently large n , there exists such a covering with not only density at most $O(n \ln n)$ but also multiplicity at most $O(n \ln n)$. In [7], Füredi and I give a new combinatorial proof of this covering result using methods from probabilistic combinatorics and discrete geometry. We use this special covering in the proof of the upper bound $c(n \ln n)5^n$ on $\chi(\mathbb{R}_p^n, 1)$.

3 Rectilinear equilateral sets in \mathbb{R}^n

Background. This is a 20-year-old problem in combinatorial geometry. Kusner [8] conjectured that the maximum size of a set whose elements are pairwise equidistant under the ℓ_1 -norm in \mathbb{R}^n is $2n$. If it is true, this would be sharp. The conjecture has been proved for $n = 3$ [3] and $n = 4$ [10]. Recently, Alon and Pudlak [1] gave an upper bound $O(n \ln n)$. These proofs are very involved and use ideas from embeddings of metric spaces, approximation theory, etc. However, in the words of the authors, their methods won't solve the full conjecture.

Results and future work. Since this problem has the same flavor as that of computing

$\chi(\mathbb{R}_1^n, 1)$, I have worked on it as well. I have made some progress on it.

I have converted the problem into a certain coloring of the sphere. I have showed that the Kusner conjecture is true if there exists an equidistant set of size $3n/2$ on the surface of the generalized-octahedron with radius 1 in \mathbb{R}^n . Related to this, I have showed that the maximum size of an equidistant set on the surface of the unit generalized-octahedron without its extreme point is at most n , and this is sharp.

I have also related this problem to the piercing (transversal) number of convex bodies in \mathbb{R}^n . I have shown that if \mathcal{F} is a family of the unit generalized-octahedrons such that every two members of \mathcal{F} intersect, then every three members of \mathcal{F} must intersect. I am working on extending this result to showing that a same \mathcal{F} with at least $2n + 1$ members must be an intersecting family. This would be enough to prove the Kusner conjecture.

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