

1 Distance graph, p -adic approach

While coloring of a metric space in high dimensions has a flavor of Combinatorial Geometry, an analogous question asked for the integer line has more of a flavor of Combinatorial Number Theory. We want to partition the integer line so that any part avoids a pair of integers whose difference belongs to a prescribed, so-called, *distance set* of positive integers.

Background. The integer *distance graph* $G(\mathbb{Z}, D)$ with distance set $D = \{d_1, d_2, \dots\}$ has the set of integers \mathbb{Z} as the vertex set and two vertices $x, y \in \mathbb{Z}$ are adjacent if and only if $|x - y| \in D$. The integer distance graphs were first systematically studied by Eggleton–Erdős–Skilton in 1985 [1, 2], and have been investigated in many ways [9, 11, 12, 14]. One of main goals in these problems is characterizing prescribed distance sets that make the corresponding distance graphs to have finite chromatic number. Ruzsa, Tuza, and Voigt [9] gave a sufficient condition for $\chi(G(\mathbb{Z}, D))$ to be finite:

Theorem 1 (Ruzsa–Tuza–Voigt [9]) *If $\inf d_{i+1}/d_i > 1$, then $\chi(G(\mathbb{Z}, D))$ is finite.*

Hence we need to investigate distance sets D having $\inf d_{i+1}/d_i = 1$ for the characterization of distance sets D with finite chromatic number.

Results. Maharaj and I approach the problem from the viewpoint of p -adic norms in [7]. We use p -adic norm to derive results on Euclidean distance graphs. For a pair of integers x and y , we write $x|y$ if x divides y , and $x \nmid y$ if x doesn't divide y . Let p be a prime number. Then any rational number x can be uniquely written in the form $x = \frac{r}{s}p^\ell$ where $\ell \in \mathbb{Z}$ and r, s are integers not divisible by p . One defines the *p -adic norm* of x by $\|x\|_p := 1/p^\ell$. This gives rise to a non-Archimedean norm on the rationals \mathbb{Q} . The completion of \mathbb{Q} with respect to this norm is denoted by \mathbb{Q}_p . It is a known fact that \mathbb{Q}_p can be identified with the set of all series $x := \sum_{i=\ell}^{\infty} a_i p^i$ where $0 \leq a_i \leq p - 1$ and $a_\ell \neq 0$. The sequence (a_i) is eventually periodic iff x belongs to \mathbb{Q} . See [6, 8] for reference.

The p -adic norm is not only an important norm in number theory but also has a natural interpretation into Euclidean norm that allows us to nicely describe the class of distance sets D with $\inf d_{i+1}/d_i = 1$. More precisely, for an integer $x > 1$, the interpretation of p -adic distance into Euclidean distance (and vice versa) is done through the product formula $|x| \prod_p \|x\|_p = 1$, where the product is over all prime numbers p , and $|\cdot|$ is the Euclidean norm. In [7], Maharaj and I have given bounds on the chromatic number, including several exact ones, that depend on the divisibility properties of the numbers d_i and that are applicable even in the case that $\inf d_{i+1}/d_i = 1$. We proved all of our results in terms of p -adic norm. Since Euclidean distance graphs have been extensively studied, we state some of the results here in terms of the Euclidean norm. For example, suppose that p_1, p_2, \dots, p_n are distinct prime numbers and D_i is a finite set of distinct non-negative powers of p_i of size $k_i := |D_i|$ for each $i = 1, 2, \dots, n$. If $D := \{aq : \text{for some } 1 \leq i \leq n, q \in D_i \text{ in which case } p_i \nmid a\}$, then $\chi(G(\mathbb{Z}, D)) = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$. If $D := \{aq_1 q_2 \dots q_t : \text{for each } 1 \leq i \leq n, q_i \in D_i \text{ and } p_i \nmid a\}$, then $\chi(G(\mathbb{Z}, D)) = \min(p_i^{k_i} : 1 \leq i \leq n)$.

One of our main results states a sufficient condition for an Euclidean distance graph $G(\mathbb{Z}, D)$ to have finite chromatic number as follows.

Theorem 2 (K.–M. [7]) *Let $D = \{d_1, d_2, \dots\}$ be a given distance set. For each prime number p , let $D(p)$ be the set of all powers p^n of p such that p^n divides d_i but p^{n+1} does not divide d_i for some i . Then*

$$\chi(G(\mathbb{Z}, D)) \leq \min(p^{|D(p)|} : p \text{ is prime}).$$

For example, if D is any set of odd numbers, then $\chi(G(\mathbb{Z}, D)) \leq 2$ since $D(2) = \{1\}$. Observe that it follows from Theorem 2 that if a distance graph $G(\mathbb{Z}, D)$ has infinite chromatic number, then arbitrarily high powers of every prime number appear as divisors of the numbers in the distance set D . Theorem 2 can be viewed as complementing the Theorem 1 of Ruzsa–Tuza–Voigt. For example, let $p_1 < p_2 < \dots$ be an enumeration of the prime numbers. Set $D = \{d_1, d_2, \dots\}$ where $d_i := (p_1 p_2 \dots p_i)^i$ for each i . Then by Theorem 1, $\chi(G(\mathbb{Z}, D))$ is finite but Theorem 2 is inconclusive. On the other hand, if D is the set of all positive integers not divisible by a fixed prime number p (so $D(p) = \{1\}$), then Theorem 2 implies that $\chi(G(\mathbb{Z}, D)) \leq p$ while Theorem 1 is inconclusive. In this sense, Theorems 1 and 2 complement each other.

Finally, we state our current strongest result as Theorem 3, which gives a conditional characterization of a distance set having finite chromatic number. Let Λ be a subset of n -tuples over nonnegative integers \mathbb{N}_0^n . We define an order $(e_1, e_2, \dots, e_n) < (e'_1, e'_2, \dots, e'_n)$ in Λ if $e_i < e'_i$ for each $1 \leq i \leq n$.

Theorem 3 (K.–M. [7]) *Let p_1, p_2, \dots, p_n be distinct prime numbers. Let $\Lambda \subset \mathbb{N}_0^n$. Define*

$$D := \{ap_1^{e_1} p_2^{e_2} \dots p_n^{e_n} : (e_1, e_2, \dots, e_n) \in \Lambda, a \in \mathbb{Z} \text{ with } p_i \nmid a \text{ for all } 1 \leq i \leq n\}. \quad (1)$$

Then the distance graph $G(\mathbb{Z}, D)$ has infinite chromatic number iff the exponent set Λ contains a strictly increasing sequence.

Future work. For a complete characterization of distance sets D having chromatic number finite, we need to drop or relax the condition of finite number of primes expressing the distance set D in Theorem 3. We believe that the characterization can be improved as follows.

Conjecture (K.–M.) Suppose that $D = \{d_1, d_2, \dots\}$. The chromatic number of Euclidean distance graph $G(\mathbb{Z}, D)$ is infinite iff some multiple of every integer appears in the set D and $\inf d_{i+1}/d_i = 1$.

To prove this conjecture concerning Euclidean distance graphs, the first step would be to prove the following conjecture that concerns p -adic distance graphs.

Conjecture (K.–M.) A p -adic distance graph $G(\mathbb{Z}, D_p)$, with a p -adic distance set D_p , has infinite chromatic number iff there exist a set of finite distinct primes p_1, p_2, \dots, p_n and a set of n -tuples Λ over nonnegative integers that consists of a strictly increasing sequence such that D_p contains a set of distances that is the set of (1) written in terms of p -adic language.

2 The persistence of a number

Background. In the sequence 679, 368, 168, 48, 32, 6, each term is the product of the decimal digits of the previous one. Neil Sloane [10] defines the *persistence* of a number as the number of steps (five in the example) before the number collapses to a single digit. The numbers with persistence 1, 2, \dots , 11 have been found, and there is no number less than 10^{50}

with persistence greater than 11. Sloane conjectured that there is a number d such that no number has persistence greater than d .

In the binary representation, the maximum persistence is 1. In the ternary representation, the second term is zero or a power of 2. It is conjectured that all powers of 2 greater than 2^{15} contain a zero when written in ternary. (The number 2^{15} consists of only 1s and 2s in ternary.) This is true up to 2^{500} . The importance of this conjecture is that the maximum persistence in the ternary representation is 3.

Sloane's general conjecture is that, for a given positive integer b , there is a number $d(b)$ such that the persistence in base b (that is, the b -ary representation) is at most $d(b)$.

Results and future work. I have worked on the conjecture of the ternary representation. Observe that if 2^n contains 12 in its ternary representation, 2^{n+1} must contain zero. I have proved that the power of 2 contains 0 or 12 in ternary. This implies that at least half of the powers of 2 contain zero in their ternary representations. This partial result is proved by considering only special ternary representations – consecutive 1s and 2s. I am working on extending the method to arbitrary appearances of 1s and 2s to complete the conjecture.

3 A variation on Van der Waerden: No consecutive monochromatic integers

to be added

4 A sequence of integers with gaps from a given set

to be added

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to be added more