

## 1 Distance graph in $p$ -adic norm

While a coloring of a metric space in high dimensions has a flavor of Combinatorial Geometry, an analogous question asked for the integer line has more of a flavor of Combinatorial Number Theory. We want to partition the integer line so that any part avoids a pair whose difference belong to a prescribed, so-called, *distance set* of positive integers.

**Background.** The integer *distance graph*  $G(\mathbb{Z}, D)$  with distance set  $D = \{d_1, d_2, \dots\}$  has the set of integers  $\mathbb{Z}$  as the vertex set and two vertices  $x, y \in \mathbb{Z}$  are adjacent if and only if  $|x - y| \in D$ . The integer distance graphs (under Euclidean norm) were first systematically studied by Eggleton–Erdős–Skilton in 1985 [12, 13], and have been investigated in many ways [50, 56, 57, 61]. One of main goals in these problems is characterizing prescribed distance sets that make the corresponding distance graphs to have finite chromatic number. Ruzsa, Tuza, and Voigt [50] gave a sufficient condition for  $\chi(G(\mathbb{Z}, D))$  to be finite:

**Theorem 1** (Ruzsa–Tuza–Voigt [50]) *If  $\inf d_{i+1}/d_i > 1$ , then  $\chi(G(\mathbb{Z}, D))$  is finite.*

Hence it is needed to investigate distance sets  $D$  having  $\inf d_{i+1}/d_i = 1$  for the characterization of distance sets  $D$  with finite chromatic number.

**Results.** Maharaj and I approaches the problem from the viewpoint of  $p$ -adic norms in [36]. For a pair of integers  $x$  and  $y$ , we write  $x|y$  if  $x$  divides  $y$ , and  $x \nmid y$  if  $x$  doesn't divide  $y$ . Let  $p$  be a prime number. Then any rational number  $x$  can be uniquely written in the form  $x = \frac{r}{s}p^\ell$  where  $\ell \in \mathbb{Z}$  and  $r, s$  are integers not divisible by  $p$ . One defines the  *$p$ -adic norm* of  $x$  by  $\|x\|_p := 1/p^\ell$ . This gives rise to a non-Archimedean norm on the rationals  $\mathbb{Q}$ .

The  $p$ -adic norm not only itself is an important norm in number theory but also has a natural interpretation into Euclidean norm that allows us to nicely describe the class of distance sets  $D$  with  $\inf d_{i+1}/d_i = 1$ . More precisely, for an integer  $x > 1$ , the interpretation of  $p$ -adic distance into Euclidean distance (and vice versa) is done through the product formula  $|x| \prod_p \|x\|_p = 1$ , where the product is over all prime numbers  $p$ , and  $|\cdot|$  is the Euclidean norm. In [36], Maharaj and I have given bounds on the chromatic number, including several exact ones, that depend on the divisibility properties of the numbers  $d_i$  and that are applicable even in the case that  $\inf d_{i+1}/d_i = 1$ .

For example, (since Euclidean distance graphs have been extensively studied, we state here in terms of the usual Euclidean norm) suppose that  $p_1, p_2, \dots, p_n$  are distinct prime numbers and  $D_i$  is a finite set of distinct non-negative powers of  $p_i$  of size  $k_i := |D_i|$  for each  $i = 1, 2, \dots, t$ . If  $D := \{aq : \text{For some } 1 \leq i \leq n, q \in D_i \text{ in which case } p_i \nmid a\}$ , then  $\chi(G(\mathbb{Z}, D)) = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$ . If  $D := \{aq_1 q_2 \dots q_t : \text{For each } 1 \leq i \leq n, q_i \in D_i \text{ and } p_i \nmid a\}$ , then  $\chi(G(\mathbb{Z}, D)) = \min(p_i^{k_i} : 1 \leq i \leq n)$ .

One of our main results states a sufficient condition for an Euclidean distance graph  $G(\mathbb{Z}, D)$  to have finite chromatic number as follows.

**Theorem 2** (K.–M. [36]) *Let  $D = \{d_1, d_2, \dots\}$  be a given distance set. For each prime number  $p$ , let  $D(p)$  be the set of all powers  $p^n$  of  $p$  such that  $p^n$  divides  $d_i$  but  $p^{n+1}$  does not*

divide  $d_i$  for some  $i$ . Then

$$\chi(G(\mathbb{Z}, D)) \leq \min\{p^{|D(p)|} : p \text{ is prime}\}.$$

For example, if  $D$  is any set of odd numbers, then  $\chi(G(\mathbb{Z}, D)) \leq 2$  since  $D(2) = \{1\}$ . Observe that it follows from Theorem 2 that if a distance graph  $G(\mathbb{Z}, D)$  has infinite chromatic number, then arbitrarily high powers of every prime number appear as divisors of the numbers in the distance set  $D$ . Theorem 2 can be viewed as complementing the Theorem 1 of Ruzsa–Tuza–Voigt. For example, let  $p_1 < p_2 < \dots$  be an enumeration of the prime numbers. Set  $D = \{d_1, d_2, \dots\}$  where  $d_i := (p_1 p_2 \dots p_i)^i$  for each  $i$ . Then by Theorem 1,  $\chi(G(\mathbb{Z}, D))$  is finite but Theorem 2 is inconclusive. On the other hand, if  $D$  is the set of all positive integers not divisible by a fixed prime number  $p$  (so  $D(p) = \{1\}$ ), then Theorem 2 implies that  $\chi(G(\mathbb{Z}, D)) \leq p$  while Theorem 1 is inconclusive. In this sense, Theorems 1 and 2 complement each other.

Finally, we state our current strongest result as Theorem 3, which gives a conditional characterization of a distance set having finite chromatic number. Let  $\Lambda$  be a subset of  $n$ -tuples over nonnegative integers  $\mathbb{N}_0^n$ . We define an order  $(e_1, e_2, \dots, e_n) < (e'_1, e'_2, \dots, e'_n)$  in  $\Lambda$  if  $e_i < e'_i$  for each  $1 \leq i \leq n$ .

**Theorem 3** (K.–M. [36]) *Let  $p_1, p_2, \dots, p_n$  be distinct prime numbers. Let  $\Lambda \subset \mathbb{N}_0^n$ . Define*

$$D := \{ap_1^{e_1} p_2^{e_2} \dots p_n^{e_n} : (e_1, e_2, \dots, e_n) \in \Lambda, a \in \mathbb{Z} \text{ with } p_i \nmid a \text{ for all } 1 \leq i \leq n\}. \quad (1)$$

*Then the distance graph  $G(\mathbb{Z}, D)$  has infinite chromatic number iff the exponent set  $\Lambda$  contains a strictly increasing sequence.*

**Future work.** For a complete characterization of distance sets  $D$  having chromatic number finite, we need to drop or relax the condition of finite number of primes expressing the distance set  $D$  in Theorem 3. We believe that the characterization can be improved as follows.

**Conjecture 1.** (K.–M.) Suppose that  $D$  is a given distance set. The chromatic number of Euclidean distance graph  $G(\mathbb{Z}, D)$  is infinite iff for every finite partition of  $D = \cup_{1 \leq j \leq k} D_j$ , there exists  $j$ ,  $1 \leq j \leq k$ , such that some multiples of every integer appear in the set  $D_j = \{d_1 < d_2 < \dots\}$  and  $\inf_{d_i \in D_j} d_{i+1}/d_i = 1$ .

To have Conjecture 1 that concerns Euclidean distance graphs, the first step would be proving the following conjecture that concerns  $p$ -adic distance graphs.

**Conjecture 2.** (K.–M) A  $p$ -adic distance graph  $G(\mathbb{Z}, D_p)$ , with a  $p$ -adic distance set  $D_p$ , has infinite chromatic number iff there exist a set of finite distinct primes  $p_1, p_2, \dots, p_n$  and a set of  $n$ -tuples  $\Lambda$  over nonnegative integers that consists of a strictly increasing sequence such that  $D_p$  contains a set of distances that is the set of (1) written in terms of  $p$ -adic language.

Another natural generalization of the distance graph problem on  $\mathbb{Z}$  is as follows. We use a specific example to illustrate the idea. Let  $K = \mathbb{Q}(i)$  where  $i^2 = -1$  and let  $R = \mathbb{Z}[i]$ . Then  $R$  is the ring of integers of  $K$ , that is,  $R$  is the integral closure of  $\mathbb{Z}$  in  $K$ . As an Abelian group,  $R$  is isomorphic to  $\mathbb{Z}^2$  and geometrically  $R$  is a lattice in the plane. A prime  $p$  of  $\mathbb{Z}$  gives rise to two  $p$ -adic norms on  $R$  if  $p \equiv 1 \pmod{4}$ , otherwise  $p$  gives rise to a single  $p$ -adic norm on  $R$  (see [4]). The product formula  $|x| \prod_p \|x\|_p = 1$  still holds, so specifying an Euclidean norm

is tantamount to specifying several  $p$ -adic norms. The same type of questions considered in this section can be asked about distance graphs on  $R$ .

In general, if  $R$  is a ring of integers in a number field  $K$ , then it is possible for  $R$  to have several inequivalent archimedean norms. An example of this would be the ring of integers of the field  $\mathbb{Q}(\sqrt[3]{2})$  which has two complex embeddings and one real embedding so there are two inequivalent Euclidean norms. It would be interesting to study how the structure of the distance graphs would change when one transitions from  $\mathbb{Z}$  to general rings of integers. It is hoped that by studying such generalisations, light would be shed on the standard distance graph problem on  $\mathbb{Z}$ . Furthermore, it is known that if  $K$  is a degree  $n$  extension of  $\mathbb{Q}$  then the ring  $R$  is a free  $\mathbb{Z}$ -module of rank  $n$ . So studying distance graphs on  $R$  is tantamount to studying a class of distance graphs on the lattice  $\mathbb{Z}^n$ . In the next section we consider a class of distance graphs on  $\mathbb{Z}^n$  with respect to the  $\ell_1$  norm.

## 2 Distance Graphs on $\mathbb{Z}^n$

**Background.** One generalization of the Hadwiger-Nelson problem is the considering of the  $n$ -dimensional real space under  $\ell_p$ -norm, which has geometric aspects. Another natural candidate is the discrete space  $\mathbb{Z}^n$ . In this case, the  $\ell_1$ -norm is the most natural metric. For a positive integer  $r$ , by  $(\mathbb{Z}^n, r)$  we mean the graph with vertex set  $\mathbb{Z}^n$  and two vertices  $x, y$  are adjacent iff  $\|x - y\|_1 = r$  where  $\|\cdot\|_1$  is the  $\ell_1$  norm throughout in this section. More generally, for  $D \subseteq \mathbb{N}$ , by  $(\mathbb{Z}^n, D)$  we mean the graph with vertex set  $\mathbb{Z}^n$  and two vertices  $x, y$  are adjacent iff  $\|x - y\|_1 \in D$ . For the case  $D = \{r : 1 \leq r \leq d - 1\}$  where  $d \leq n$ , Louge[43] use results from coding theory to give various lower bounds of the chromatic number. Some of them are

**Theorem 4** *Let  $G = (\mathbb{Z}^n, 1 \leq r \leq d - 1)$ . Then*

$$\chi(G) \geq \frac{2^n}{A(n, d)} \quad (2)$$

and

**Theorem 5** *There exists a sequence of positive integers  $n_1 < n_2 < \dots$  and  $d_1, d_2, \dots$  such that  $d_i \leq n_i$  for all  $i$  and  $\frac{d_i}{n_i} \rightarrow \delta$  with*

$$\lim_{i \rightarrow \infty} \frac{\log_2 \chi(G_i)}{n_i} \geq 1 - \alpha_2(\delta)$$

where  $G_i = (\mathbb{Z}^{n_i}, 1 \leq r \leq d_i - 1)$ .

Note that the subscript  $p$  of  $\mathbb{Q}$  indicates that the  $\ell_p$  norm is being used.

**Results.** In [21], Füredi and I investigate  $(\mathbb{Z}^n, r)$ . Since the graph  $(\mathbb{Z}_1^n, r)$  is bipartite when  $r$  is odd, we study  $\chi(\mathbb{Z}_1^n, r)$  for even  $r$ . We prove that  $\chi(\mathbb{Z}_1^n, 2) = 2n$  for all  $n$ , but  $\chi(\mathbb{Z}^n, r) \geq 2n + 1$  for  $n \geq 3$  and even  $r \geq 4$ . In addition, we give an exponential lower bound  $(1.139)^n$  by constructing an appropriate finite subgraph, again using the Frankl-Wilson inequality. Since  $(\mathbb{Z}_1^n, r)$  is a subgraph of  $(\mathbb{R}_1^n, r)$ , we have  $\chi(\mathbb{Z}_1^n, r) \leq \chi(\mathbb{R}_1^n, r) = \chi(\mathbb{R}_1^n, 1)$ , so the upper bounds for  $\chi(\mathbb{R}_1^n, 1)$  in Section ?? hold here. For small values of  $r$ , we have a more

useful upper bound,  $3r^{n-2}$  for all  $n$  and all even  $r \geq 4$ , which follows from a recurrence that extends a coloring on a selected hyperplane to the whole space.

**Work in progress.** In [1], the case  $D = \{t : 1 \leq r < d_1 \text{ or } d_2 < r \leq n\}$  where  $d_1 \leq d_2$  has been investigated, and it is being typed now. This turns out to involve more interesting stuff from coding theory, namely codes in spherical caps in Euclidean space.

**Future work.** We want to determine the clique number in the induced subgraph on the subset  $\{0, 1\}^n$ . The clique number not only serves the lower bound of  $\chi(\mathbb{Z}^n, r)$  for a fixed constant  $r$  but also is important in coding theory to obtain information on the size of optimal codes with specified minimum distances.

One of the main goals is to prove analogues of Theorems 4 and 5 for more general distance sets. From the proof of Theorem 4, these results depend only on the subgraph induced on  $\{0, 1\}^n$ . One would expect better results if one considers the induced subgraph on  $V' := \{0, 1, 2, \dots, q\}^n$ . However, the distance no longer coincides with the Hamming distance. We would have to consider the following problem: find upper bounds on the size of a set  $C \subset V'$  such that  $\|x - y\|_1 \in \{0, 1, \dots, d\}$ . If  $\gamma(q, d)$  is such an upper bound then immediately we get that for  $1 \leq d \leq n$ ,  $\chi(\mathbb{Z}^n, 1 \leq r \leq d) \geq q^n / \gamma(q, d)$ . It is expected that the techniques used in the papers [2, 3] would need to be adapted. We also expect better asymptotic results than those in Theorem 5. In order to adapt the results to graphs  $(\mathbb{Z}^n, D)$  to more general distance sets  $D$ , it would again be necessary to adapt the coding theoretic results appropriately. For example, if  $D$  is a non-empty subset of  $\{1, \dots, n - 1\}$  we would have to find good upper bounds on the size of a largest subset  $C$  of  $\{0, 1\}^n$  with the property that  $d(x, y) \in D$  for all distinct  $x, y \in C$ . If the set  $D$  contains integers greater than  $n - 1$ , then we would have to consider subsets  $\mathcal{C}$  of the set  $\{0, 1, 2, \dots, q - 1\}^n$  where  $q$  is the smallest integer such that  $(q - 1) \cdot n \geq \max D$  and find good upper bounds on a largest subset  $\mathcal{C}$  of  $\{0, 1, 2, \dots, q - 1\}^n$  with the property that  $\|x - y\|_1 \in D$  for all distinct  $x, y \in \mathcal{C}$ . In both cases it is expected that the techniques of the papers [2, 3] to be adapted to solve these problems.

### 3 Distance graphs on the integer line

For a complete characterization of Euclidean distance graphs on the integer line described in Section 1, Maharaj and I have pursued further on the problem using distribution of fractional part of real numbers. We obtain a bit stronger result than in Section 1.

### 4 On largeness and accessibility

Let  $D$  be a prescribed subset of the integer set  $\mathbb{Z}$ . If any  $r$ -coloring of  $\mathbb{Z}^+ \cup \{0\}$  yields an arbitrarily long monochromatic arithmetic progression whose common difference  $d$  belongs to  $D$ , we say that  $D$  is *r-large*. If any  $r$ -coloring of  $\mathbb{Z}^+ \cup \{0\}$  yields an arbitrarily long monochromatic sequence of distinct integers whose gaps belong to  $D$ , the set  $D$  is called *r-accessible*. Landman and Robertson ask to determine the largest number  $r$  such that the prime set  $P$  is *r-accessible*, and the same question for its translate  $P + c$ . It turns out difficult to determine whether  $P$  is 2-accessible or not. In this paper, we investigate the largeness and the accessibility of  $D_m$  and  $D_m + c$  where  $D_m$  is all the integers that is coprime to a fixed

integer  $m \geq 2$  and  $c \geq 1$ . We give bounds on those, and in particular, we show that  $D_m + c$  is  $r$ -large, and consequently  $r$ -accessible, for all  $r \geq 1$  when  $\gcd(m, c) = 1$ .

## 5 The persistence of a number

**Background.** In the sequence 679, 368, 168, 48, 32, 6, each term is the product of the decimal digits of the previous one. Neil Sloane [?] defines the *persistence* of a number as the number of steps (five in the example) before the number collapses to a single digit. The numbers with persistence 1, 2,  $\dots$ , 11 have been found, and there is no number less than  $10^{50}$  with persistence greater than 11. Sloane conjectured that there is a number  $d$  such that no number has persistence greater than  $d$ .

In the binary representation, the maximum persistence is 1. In the ternary representation, the second term is zero or a power of 2. It is conjectured that all powers of 2 greater than  $2^{15}$  contain a zero when written in ternary. (The number  $2^{15}$  consists of only 1s and 2s in ternary.) This is true up to  $2^{500}$ . The importance of this conjecture is that the maximum persistence in the ternary representation is 3.

Sloane's general conjecture is that, for a given positive integer  $b$ , there is a number  $d(b)$  such that the persistence in base  $b$  (that is, the  $b$ -ary representation) is at most  $d(b)$ .

**Results and future work.** I have worked on the conjecture of the ternary representation. Observe that if  $2^n$  contains 12 in its ternary representation,  $2^{n+1}$  must contain zero. I have proved that the power of 2 contains 0 or 12 in ternary. This implies that at least half of the powers of 2 contain zero in their ternary representations. This partial result is proved by considering only special ternary representations – consecutive 1s and 2s. I am working on extending the method to arbitrary appearances of 1s and 2s to complete the conjecture.

## 6 A variation on Van der Waerden: No consecutive monochromatic integers

*to be added*

## 7 A sequence of integers with gaps from a given set

*to be added*

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