

Extremal graph theory is, broadly speaking, the study of relations between various graph invariants, such as order, size, connectivity, minimum/maximum degree, chromatic number, etc., and the values of these invariants that ensure that the graph has certain properties. Since the first major result by Turan in 1941, numerous mathematicians have contributed to make this a vibrant and deep subject. Among all its topics, graph coloring is the most applicable and widely studied. Typically, a graph models conflicts, and a good coloring ensures partitions into parts with no conflicts. See Bollobás [2] and Jensen and Toft [8] for an overview.

## 1 $L(2, 1)$ -labeling for graphs

**Background.** In ordinary graph coloring, adjacent vertices must be given different colors, and the actual values of the colors used are irrelevant. However, in many applications, it is also important to separate labels on vertices at farther distances, where the labels used have some numerical meaning. A natural problem of this type is the channel assignment problem, where channels (non-negative integers) are assigned to each radio transmitter (vertex) so that interfering (adjacent) transmitters get channels that are far apart. F.S. Roberts proposed a variation of the channel assignment problem, which Griggs and Yeh [7] introduced in 1992 and called the  $L(2, 1)$ -labeling problem. Keeping the radio transmitter analogy in mind, vertices in a graph need to be labeled such that “close” vertices (at distance 2) get different labels while “very close” vertices (at distance 1) get labels that are farther apart. More precisely, for a given graph  $G$ , a mapping  $f : V(G) \rightarrow \mathbb{N} \cup \{0\}$  is called an  $L(2, 1)$ -labeling if  $|f(u) - f(v)| \geq 2$  for each edge  $uv$  of  $G$  and  $|f(u) - f(v)| \geq 1$  for each pair  $u, v \in V(G)$  at distance 2 apart. The  $L(2, 1)$ -labeling number of  $G$ , denoted by  $\lambda(G)$ , is the smallest number  $t$  such that  $G$  has a  $L(2, 1)$ -labeling that does not use any label greater than  $t$ .

As in the case of chromatic number of graphs, the maximum degree of a graph,  $\Delta(G)$ , is a natural candidate for bounding  $\lambda(G)$ . The obvious lower bound for  $\lambda$  is  $\Delta + 1$ , which holds with equality for the star  $K_{1, \lambda}$ . A greedy labeling (as shown in [7]) gives  $\lambda(G) \leq \Delta^2 + 2\Delta$ . This upper bound was improved to  $\Delta^2 + \Delta$  in [4]. Griggs and Yeh [7] conjectured that for every graph  $G$ ,  $\lambda(G) \leq \Delta^2(G)$ . This has been a motivating problem for research in this field, and some results are known. Note that it is enough to consider connected, regular graphs. Tight bounds have been obtained for special classes of graphs like paths, cycles, wheels, complete  $k$ -partite graphs and graphs with diameter 2 [7], trees [4, 7], etc. Some bounds have also been obtained for various other graph families like chordal graphs and unit interval graphs [14], hypercubes [7, 9, 16], and planar graphs [9]. See [1] for a wide ranging survey including algorithms, complexity and applications to communication networks. However, the core of the conjecture remains wide open – even for 3-regular graphs.

**Results and future work.** I proved the Griggs–Yeh Conjecture for 3-regular Hamiltonian graphs [11]. The proof is rather intricate, and requires the study of structural properties of the involved graphs. It starts by pre-labeling  $G$  to produce a graph  $H$  of ‘badly’ labeled pairs of vertices, and then it uses the structure of  $H$  to reduce finding  $L(2, 1)$ -labeling of  $G$  to finding an ordinary coloring of  $H$  satisfying some additional constraints.

I also studied  $L(2, 1)$ -labeling of two special graphs, the incidence graph of the projective

plane of order  $q$ , and the Kneser graph. These graphs are very interesting and frequently occur in a variety of problems. Since they have a rich structure and it is not trivial to analyze most graph parameters for them, doing so can lead to insights into the more general problem.

The Kneser graph  $K(m, k)$  is the disjointness graph on the  $k$ -subsets of  $\{1, 2, \dots, m\}$ . For  $K(2k+1, k)$ , in [12] I showed that  $\lambda(G) \leq 4k+2$ . Here, the  $L(2, 1)$ -labeling is obtained from a classification of structures between and within the color classes of a special vertex coloring.

For the incidence graph  $G$  of the projective plane  $PG(2, q)$ , Füredi and I [6] show that  $\lambda(G) = q^2 + q = \Delta^2 - \Delta$ . (The problem was also studied in [9]) To prove this result, we considered packing bipartite graphs into a complete bipartite graph and proved a sufficient condition for such a packing, which is analogous to the result of Sauer and Spencer [15] for packing graphs into a complete graph. For given bipartite graphs  $G_1$  and  $G_2$  with bipartitions  $X_1, Y_1$  and  $X_2, Y_2$ , respectively, a *packing of  $G_1$  and  $G_2$  into  $K_{m,n}$*  maps  $X_1 \rightarrow [m], Y_1 \rightarrow [n]$  and  $X_2 \rightarrow [m], Y_2 \rightarrow [n]$  injectively such that  $E(G_1) \cap E(G_2) = \emptyset$ . We showed that  $2\Delta(G_1)\Delta(G_2) < 1 + \max\{m, n\}$  is a sufficient condition for such a packing.

The result on 3-regular Hamiltonian graphs is the first significant progress towards the Griggs and Yeh conjecture in the last few years. However, the extra condition of Hamiltonicity needs to be removed (to complete the proof for 3-regular graphs). A Hamiltonian cycle can be thought as a 2-factor consisting of one cycle. I am currently working on this with D. West by considering 3-regular graphs with 2-factors consisting of arbitrarily many cycles. Note that every 2-edge-connected 3 regular graph has a 2-factor.

We are also working on extending the ideas from the incidence graph of  $PG(2, q)$  to a more general class of bipartite graphs. Füredi and I have succeeded in classifying the case  $\Delta = 3$  and are pursuing other cases.

## 2 Security Number

**Background.** A *defensive alliance* in a graph  $G = (V, E)$  is a subset  $S$  of  $V$  with the property that every vertex  $x \in S$  has as many neighbors, including  $x$  itself, in  $S$  as those in  $V \setminus S$  [13]. The various concepts of alliances in graphs are motivated from a security issue that whether defenders of  $x$  in  $S$  can defeat the attackers of  $x$  in  $V \setminus S$ . Recently, Brigham, Dutton, and Hedetniemi [3] introduced a global concept of an alliance: A subset  $S \subset V$  is a *secure set* if every subset  $U \subset S$  has as many neighbors, including  $U$ , in  $S$  as those in  $V \setminus S$ . We seek the parameter  $\rho(G)$ , the *security number* of  $G$ , that is the minimum size of a secure set in  $G$ .

It is easy to construct a graph having  $\rho(G)$  arbitrarily small compared to  $n(G)$ , the number of vertices of  $G$ , in which the connectivity of the graph is low. Hence we are interested in how large  $\rho(G)$  can be in terms of  $n(G)$ , and consider graphs with high connectivity. For instance, a complete graph of order  $n$  has the security number  $\lceil n/2 \rceil$ . Brigham, Dutton, and Hedetniemi [3] asked whether  $\lceil n/2 \rceil$  is an upper bound for  $\rho(G)$  for every graph  $G$ . In particular, Brigham (personal communication) asked whether the security number of Kneser graphs  $K(m, k)$  is at most  $\lceil n/2 \rceil$  where  $n = \binom{m}{k}$ .

**Results and future work.** I have proved that  $\rho(K(m, 2)) = \lceil \frac{n+1}{2} \rceil$  (independently by Dutton–Lee–Brigham [5]). This is a negative answer to their question and makes the problem more interesting. In my proof of  $\rho(K(m, 2)) = \lceil \frac{n+1}{2} \rceil$ , I use the language of matching and covering numbers. I am working on extending this idea to obtain the security number for the

whole class of Kneser graphs.

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