MATH 1634  \( \epsilon - \delta \) Game; Compact and Complete Version

For a given \( \lim_{x \to a} f(x) \),

**Player A** (Student) Asserts that a certain number \( L \) is the limit:

\[
\lim_{x \to a} f(x) = L.
\]

**Player B** (Teacher) Challenges this assertion by giving Player A a specific value for \( \epsilon > 0 \).

**Player A** Must respond to the challenge by coming up with a value of \( \delta > 0 \)

such that

\[
|f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.
\]

**CONCLUSION.**  *Exactly one of two options:*

- If Player A can always find a value of \( \delta > 0 \) that works, then he wins, and the limit converges to \( L \).

- If Player B can give a specific value of \( \epsilon > 0 \) for which Player A cannot respond adequately, then Player B wins, and we conclude that the limit does not converge to \( L \).
Examples of “one value” for $\delta$

**Example 1** $\lim_{x \to 3}(2x) \text{ is given.}$

**Player A** (Student) Asserts that $\lim_{x \to 3}(2x) = \rule{2cm}{0.1pt}.$

**Player B** Challenges this assertion by giving Player A

$\epsilon > 0.$

**Player A** Come up with the value of

$$\delta = \rule{2cm}{0.1pt} > 0$$

that yields

$$|2x - 6| < \rule{2cm}{0.1pt} \text{ whenever } |x - 3| < \rule{2cm}{0.1pt}.$$

**Proof.**

$$|2x - 6| < \rule{2cm}{0.1pt} \iff |2(x - 3)| < \rule{2cm}{0.1pt} \iff |x - 3| < \rule{2cm}{0.1pt}.$$

**CONCLUSION:** For an arbitrary constant \rule{2cm}{0.1pt}, since Player A can always find a value of

\rule{2cm}{0.1pt}

that works, he wins, and the limit converges to \rule{2cm}{0.1pt}. 

2
Example 2 \( \lim_{x \to 1} (x + 1) \) is given.

**Player A** (Student) Asserts that \( \lim_{x \to 1} (x + 1) = \) _______.

**Player B** Challenges this assertion by giving Player A \( \epsilon > 0 \).

**Player A** Come up with the value of
\[
\delta = \quad > 0
\]
that yields
\[
| (x + 1) - 2 | < \quad \text{ whenever } | x - 1 | < \quad .
\]

**Proof.**
\[
| (x + 1) - 2 | < \quad \Leftrightarrow | x - 1 | < \quad .
\]

**CONCLUSION:** For an arbitrary constant _______, since Player A can always find a value of _______
that works, he wins, and the limit converges to _______.

3
Example 3 \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \) is given.

**Player A** (Student) Asserts that \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \) ________.

**Player B** Challenges this assertion by giving Player A 

\( \epsilon > 0 \).

**Player A** Come up with the value of

\( \delta = \) ________ > 0

that yields

\[
\left| \frac{x^2 - 1}{x - 1} - 2 \right| < \) ________ whenever \( |x - 1| < \) ________ except for \( x = 1 \).

**Proof.**

\[
\left| \frac{x^2 - 1}{x - 1} - 2 \right| < \) ________ \( \iff \left| \frac{(x + 1)(x - 1)}{x - 1} - 2 \right| < \) ________
\( \iff |(x + 1) - 2| < \) ________ since \( x \neq 1 \)
\( \iff |x - 1| < \) ________.

**CONCLUSION:** For an arbitrary constant ________, since Player A can always find a value of ________ that works, he wins, and the limit converges to ________.
Example 4 \( \lim_{x \to 3} (4x - 5) \) is given.

**Player A** Asserts that \( \lim_{x \to 3} (4x - 5) = \) ______.

**Player B** Challenges this assertion by giving Player A

\[ \epsilon > 0. \]

**Player A** Come up with the value of

\[ \delta = \) \] ______ > 0

that yields

__________________________ whenever ____________________.

Proof.

\[ \leftrightarrow \]

\[ \leftrightarrow \]

\[ \leftrightarrow \]

: 

CONCLUSION: For an arbitrary constant ______, since Player A can always find a value of

__________________

that works, he wins, and the limit converges to ______.
Example 5  \( \lim_{x \to 0^+} \sqrt{x} \) is given.

**Player A** (Student) Asserts that \( \lim_{x \to 0^+} \sqrt{x} = \) ______.

**Player B** Challenges this assertion by giving Player A 

\( \epsilon > 0 \).

**Player A** Come up with the value of 

\[ \delta = \ldots > 0 \]

that yields

________________________ whenever ____________________.

**Proof.**

\[ \quad \]

\[ \quad \]

\[ \quad \]

\[ \quad \]

\[ \quad \]

**CONCLUSION:** For an arbitrary constant ______, since Player A can always find a value of


that works, he wins, and the limit converges to ______.
Example 6 \( \lim_{x \to 0} x^2 \) is given.

**Player A** (Student) Asserts that \( \lim_{x \to 0} x^2 = \) ________.

**Player B** Challenges this assertion by giving Player A \( \epsilon > 0 \).

**Player A** Come up with the value of

\[ \delta = \quad \] > 0

that yields

____________________ whenever ____________________.

**Proof.**

\[ \leftrightarrow \]

\[ \leftrightarrow \]

\[ \leftrightarrow \]

\[ : \]

**CONCLUSION:** For an arbitrary constant ________, since Player A can always find a value of

___________

that works, he wins, and the limit converges to ________.
Example 7 \( \lim_{x \to 3} x^2 \) is given.

**Player A** (Student) Asserts that \( \lim_{x \to 3} x^2 = \) ______.

**Player B** Challenges this assertion by giving Player A \( \epsilon > 0 \).

**Player A** Come up with the value of

\[
\delta = \text{________} > 0
\]

that yields

\[
\text{________} \quad \text{whenever} \quad \text{________}.
\]

Proof.

**CONCLUSION:** For an arbitrary constant ______, since Player A can always find a value of

\[
\delta := \text{_____} > 0 \quad \text{(in terms of} \quad \epsilon)
\]

that works, he wins, and the limit converges to ______.
So far, Player A (Students) has always won. Here is an example that Player B (Teacher) wins.

Example of no choice for $\delta > 0$ for a given $\epsilon > 0$

Example 8. A function $f(x)$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

What is $\lim_{x \to 0} f(x)$?

ROUND 1.

Player A (Student) Asserts that $\lim_{x \to 0} f(x) = 1$.

Player B (Teacher) Challenges with $\epsilon = 0.1$

Player A. For every $\delta > 0$ that Player A suggests, we have

$$|f(x) - 1| \geq 0.1 \text{ when } 0 < |x - 0| < \delta \text{ if } x \text{ is irrational.}$$

Note that there are finitely many irrational numbers $x$ in $0 < |x - 0| < \delta$.

Proof. If $x$ is irrational, then $|f(x) - 1| = |0 - 1| = 1 \geq 0.1$ even when $0 < |x - 0| < \delta$.

CONCLUSION: Player B gives a specific value of $\epsilon = 0.1 > 0$ for which Player A cannot respond adequately, then Player B wins, and we conclude that the limit does not converge to 1.
ROUND 2.

**Player A** (Student) Asserts that \( \lim_{x \to 0} f(x) = 0. \)

**Player B** (Teacher) Challenges with \( \epsilon = 0.001 \)

**Player A** For every \( \delta > 0 \)

that Player A suggests, we have

\[ |f(x) - 0| \geq 0.001 \quad \text{when } 0 < |x - 0| < \delta \quad \text{if } x \text{ is rational.} \]

Note that there are finitely many rational numbers \( x \) in \( 0 < |x - 0| < \delta. \)

**Proof.** If \( x \) is rational, \( |f(x) - 0| = |1 - 0| = 1 \geq 0.001 \) even when \( 0 < |x - 0| < \delta. \)

CONCLUSION: Player B gives a specific value of \( \epsilon = 0.001 > 0 \) for which Player A cannot respond adequately, then Player B wins, and we conclude that the limit does not converge to 0.