

Research Statement

Jeong-Hyun Kang

1 Introduction

My research interests lie in Discrete Mathematics, especially Combinatorics, Graph Theory, Combinatorial Geometry, and Combinatorial Number Theory. For me, the most exciting aspect of working in discrete mathematics is the prevalence of combinatorial problems in various fields of mathematics and various applications to Computer Science and real life problems such as building transmitters in a town, assignment of radio channels, scheduling meetings, etc. Graph theory and combinatorics provide a unifying abstract structure for these problems. At the same time, significant progress in combinatorial problems requires application of tools/methods from a variety of fields like probability, geometry, algebra, number theory, etc. I have experienced both these aspects of research in combinatorics through my work on geometric combinatorial problems and number theoretic combinatorial problems: my proofs employ ideas from discrete Geometry, probability, asymptotic analysis, and number theory. [20, 21, 34]. In the long term, I will continue to deepen and broaden my understanding of fields like geometry, algebra, number theory, and probability, as well as combinatorics, so that I can make contributions across a wide range of topics which lie in the intersection of these fields with combinatorics.

In this statement, I will discuss my research and results to date and then briefly describe other topics that I have worked on or intend to in the near future.

2 Coloring metric spaces

Combinatorial geometry is the study of combinatorial properties of fundamental geometric objects, whose origins go back to antiquity. It has come into maturity in the last century through the seminal works of O. Helly, K. Borsuk, P. Erdős, H. Hadwiger, L. Fejes Tóth, B. Grünbaum and many other excellent mathematicians who initiated new combinatorial approaches to classical questions studied by Newton, Gauss, Minkowski, and Hilbert, as well as new areas of investigation. The textbooks by Matousek [42] and, Pach and Agarwal [45] provide an overview of the topics and methods.

Background. A fundamental approach to studying the combinatorial structure of a metric space is: If we partition a space into a small number of parts (i.e., color its points), at least one of these parts must contain certain “unavoidable” point configurations. In the most basic case, the configuration consists of a pair of points at a given distance (which can usually be scaled to one). The first such question, known as the Hadwiger–Nelson problem, was: What is the minimum number of colors needed for coloring the plane so that no two points at unit distance receive the same color? The answer is known to be between 4 and 7, with no improvement in the last 50 years. The same question can be asked for any metric space in high dimensions. This has turned out to be a challenging problem with very little progress made on it. See Pach [44] for a short survey. We consider ℓ_p -spaces and investigate $\chi(\mathbb{R}_p^n, 1)$, i.e., the chromatic number of the unit distance graph on \mathbb{R}^n under the ℓ_p -norm. Another natural candidate is the discrete space \mathbb{Z}^n . In this case, the ℓ_1 -norm is the most natural metric and, since distances cannot be scaled to one, we consider the graph (\mathbb{Z}_1^n, r) on \mathbb{Z}^n with two points adjacent if their ℓ_1 -distance is a fixed integer r . Almost nothing is known about these chromatic numbers, with the best results for the Euclidean space ($p = 2$), $(1.2 + o(1))^n \leq \chi(\mathbb{R}_2^n, 1) \leq (3 + o(1))^n$. The lower bound was given by Frankl and Wilson [18],

and the upper bound by Larman and Rogers [40] in 1981 and 1972, respectively. Other than this, only weak bounds on $\chi(\mathbb{R}_2^3, 1)$ are known. The subgraph induced by rational points has also been investigated in small dimensions: $\chi(\mathbb{Q}_2^2, 1) = 2$ (Woodall [55]), $\chi(\mathbb{Q}_2^3, 1) = 2$ and $\chi(\mathbb{Q}_2^4, 1) = 4$ (Benda and Perles [5]) are the main results. Morayne [43] used Fermat's Last Theorem to observe that $\chi(\mathbb{Q}_p^2, 1) = 2$ for $p \geq 3$.

Results. In [20, 21, 30], Füredi and I investigate $\chi(\mathbb{R}_p^n, 1)$ for general $p \geq 1$, by studying certain fundamental geometric ideas, namely, partitions of \mathbb{R}^n , sphere packing in \mathbb{R}^n , and covering of \mathbb{R}^n by translates of a convex body, which respectively led to the upper bounds, $\sqrt{p/(2\pi n)}(5(ep)^{1/p})^n$ (for all n), 9^n (for all n), and $c(n \ln n)5^n$ (for large n). The third upper bound requires a covering with special characteristics, described in the following section. We use an intersection inequality of Frankl and Wilson [18] to construct a finite subgraph of $(\mathbb{R}_p^n, 1)$ with small independence number and, consequently, large chromatic number, which gives a lower bound of $(1.139)^n$ for all p .

Since the graph (\mathbb{Z}_1^n, r) is bipartite when r is odd, we study $\chi(\mathbb{Z}_1^n, r)$ for even r . In [20], Füredi and I prove that $\chi(\mathbb{Z}_1^n, 2) = 2n$ for all n , but $\chi(\mathbb{Z}_1^n, r) \geq 2n + 1$ for $n \geq 3$ and even $r \geq 4$. In addition, we give an exponential lower bound $(1.139)^n$ by constructing an appropriate finite subgraph, again using the Frankl-Wilson inequality. Since (\mathbb{Z}_1^n, r) is a subgraph of (\mathbb{R}_1^n, r) , we have $\chi(\mathbb{Z}_1^n, r) \leq \chi(\mathbb{R}_1^n, r) = \chi(\mathbb{R}_1^n, 1)$, so the upper bounds for $\chi(\mathbb{R}_1^n, 1)$ hold here. For small values of r , we have a more useful upper bound, $3r^{n-2}$ for all n and all even $r \geq 4$, which follows from a recurrence that extends a coloring on a selected hyperplane to the whole space.

Future work. To improve the upper bounds, the first step could be to combine the packing and covering arguments used in [30] and [21]. A fundamental improvement might depend on a better understanding of the coloring of the $(n - 1)$ -dimensional sphere of radius a as a subgraph of \mathbb{R}^n . Except for a lower bound of n by Lovász (see [44]) and some bounds for small dimensions, almost nothing is known; we plan to further investigate the problem.

The exponential lower bound follows from an application of the Frankl-Wilson Inequality to a carefully defined subset of $\{0, 1\}^n$ under scaling. We try to improve the lower bound, possibly by considering a bigger subspace like $\{-1, 0, 1\}^n$.

We also want to determine the clique number in the induced subgraph on the subset $\{0, 1\}^n$. The clique number not only serves the lower bound of $\chi(\mathbb{Z}^n, r)$ for a fixed constant r but also is important in coding theory to obtain information on the size of optimal codes with specified minimum distances.

3 Covering Euclidean n -space

Roughly speaking, the *density* of a covering of \mathbb{R}^n is the average number of bodies needed to cover a given unit space. The density of a cover measures how good the cover is. We seek to minimize the density of a collection \mathcal{C} of convex bodies that covers \mathbb{R}^n .

Rogers [46] proved that, for a given closed convex body C in the n -dimensional Euclidean space, where $n \geq 3$, there is a covering for \mathbb{R}^n by translates of C with density $O(n \ln n)$. However, low density does not imply low multiplicity of the covering, where *multiplicity* is the number of copies of $C \in \mathcal{C}$ containing each point. Even though the global density of a covering is low, there can exist local clusters of high multiplicity. Later, Erdős and Rogers [14] showed that, for sufficiently large n , there exists such a covering with not only density at most $O(n \ln n)$ but also multiplicity at most $O(n \ln n)$. In [21], Füredi and I give a new combinatorial proof of this covering result using methods and tools from probabilistic combinatorics and discrete geometry, and asymptotic analysis, such as Lovasz Local Lemma and a deep theorem on volume ratio by K. Ball. We use this special covering in the proof of the upper bound $c(n \ln n)5^n$ on $\chi(\mathbb{R}_p^n, 1)$.

4 Distance graph, p -adic approach

While coloring of a metric space in high dimensions has a flavor of Combinatorial Geometry, an analogous question asked for the integer line has more of a flavor of Combinatorial Number Theory. We want to partition the integer line so that any part avoids a pair of integers whose difference belongs to a prescribed, so-called, *distance set* of positive integers.

Background. The integer *distance graph* $G(\mathbb{Z}, D)$ with distance set $D = \{d_1, d_2, \dots\}$ has the set of integers \mathbb{Z} as the vertex set and two vertices $x, y \in \mathbb{Z}$ are adjacent if and only if $|x - y| \in D$. The integer distance graphs were first systematically studied by Eggleton–Erdős–Skilton in 1985 [11, 12], and have been investigated in many ways [47, 52, 53, 57]. One of main goals in these problems is characterizing prescribed distance sets that make the corresponding distance graphs to have finite chromatic number. Ruzsa, Tuza, and Voigt [47] gave a sufficient condition for $\chi(G(\mathbb{Z}, D))$ to be finite:

Theorem 1 (Ruzsa–Tuza–Voigt [47]) *If $\inf d_{i+1}/d_i > 1$, then $\chi(G(\mathbb{Z}, D))$ is finite.*

Hence we need to investigate distance sets D having $\inf d_{i+1}/d_i = 1$ for the characterization of distance sets D with finite chromatic number.

Results. Maharaj and I approach the problem from the viewpoint of p -adic norms in [34]. We use p -adic norm to derive results on Euclidean distance graphs. For a pair of integers x and y , we write $x|y$ if x divides y , and $x \nmid y$ if x doesn't divide y . Let p be a prime number. Then any rational number x can be uniquely written in the form $x = \frac{r}{s}p^\ell$ where $\ell \in \mathbb{Z}$ and r, s are integers not divisible by p . One defines the *p -adic norm* of x by $\|x\|_p := 1/p^\ell$. This gives rise to a non-Archimedean norm on the rationals \mathbb{Q} . The completion of \mathbb{Q} with respect to this norm is denoted by \mathbb{Q}_p . It is a known fact that \mathbb{Q}_p can be identified with the set of all series $x := \sum_{i=\ell}^{\infty} a_i p^i$ where $0 \leq a_i \leq p - 1$ and $a_\ell \neq 0$. The sequence (a_i) is eventually periodic iff x belongs to \mathbb{Q} . See [23, 35] for reference.

The p -adic norm is not only an important norm in number theory but also has a natural interpretation into Euclidean norm that allows us to nicely describe the class of distance sets D with $\inf d_{i+1}/d_i = 1$. More precisely, for an integer $x > 1$, the interpretation of p -adic distance into Euclidean distance (and vice versa) is done through the product formula $|x| \prod_p \|x\|_p = 1$, where the product is over all prime numbers p , and $|\cdot|$ is the Euclidean norm. In [34], Maharaj and I have given bounds on the chromatic number, including several exact ones, that depend on the divisibility properties of the numbers d_i and that are applicable even in the case that $\inf d_{i+1}/d_i = 1$. We proved all of our results in terms of p -adic norm. Since Euclidean distance graphs have been extensively studied, we state some of the results here in terms of the Euclidean norm. For example, suppose that p_1, p_2, \dots, p_n are distinct prime numbers and D_i is a finite set of distinct non-negative powers of p_i of size $k_i := |D_i|$ for each $i = 1, 2, \dots, n$. If $D := \{aq : \text{for some } 1 \leq i \leq n, q \in D_i \text{ in which case } p_i \nmid a\}$, then $\chi(G(\mathbb{Z}, D)) = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$. If $D := \{aq_1 q_2 \dots q_t : \text{for each } 1 \leq i \leq n, q_i \in D_i \text{ and } p_i \nmid a\}$, then $\chi(G(\mathbb{Z}, D)) = \min(p_i^{k_i} : 1 \leq i \leq n)$.

One of our main results states a sufficient condition for an Euclidean distance graph $G(\mathbb{Z}, D)$ to have finite chromatic number as follows.

Theorem 2 (K.–M. [34]) *Let $D = \{d_1, d_2, \dots\}$ be a given distance set. For each prime number p , let $D(p)$ be the set of all powers p^n of p such that p^n divides d_i but p^{n+1} does not divide d_i for some i . Then*

$$\chi(G(\mathbb{Z}, D)) \leq \min(p^{|D(p)|} : p \text{ is prime}).$$

For example, if D is any set of odd numbers, then $\chi(G(\mathbb{Z}, D)) \leq 2$ since $D(2) = \{1\}$. Observe that it follows from Theorem 2 that if a distance graph $G(\mathbb{Z}, D)$ has infinite chromatic number, then

arbitrarily high powers of every prime number appear as divisors of the numbers in the distance set D . Theorem 2 can be viewed as complementing the Theorem 1 of Ruzsa–Tuza–Voigt. For example, let $p_1 < p_2 < \dots$ be an enumeration of the prime numbers. Set $D = \{d_1, d_2, \dots\}$ where $d_i := (p_1 p_2 \dots p_i)^i$ for each i . Then by Theorem 1, $\chi(G(\mathbb{Z}, D))$ is finite but Theorem 2 is inconclusive. On the other hand, if D is the set of all positive integers not divisible by a fixed prime number p (so $D(p) = \{1\}$), then Theorem 2 implies that $\chi(G(\mathbb{Z}, D)) \leq p$ while Theorem 1 is inconclusive. In this sense, Theorems 1 and 2 complement each other.

Finally, we state our current strongest result as Theorem 3, which gives a conditional characterization of a distance set having finite chromatic number. Let Λ be a subset of n -tuples over nonnegative integers \mathbb{N}_0^n . We define an order $(e_1, e_2, \dots, e_n) < (e'_1, e'_2, \dots, e'_n)$ in Λ if $e_i < e'_i$ for each $1 \leq i \leq n$.

Theorem 3 (K.–M. [34]) *Let p_1, p_2, \dots, p_n be distinct prime numbers. Let $\Lambda \subset \mathbb{N}_0^n$. Define*

$$D := \{ap_1^{e_1} p_2^{e_2} \dots p_n^{e_n} : (e_1, e_2, \dots, e_n) \in \Lambda, a \in \mathbb{Z} \text{ with } p_i \nmid a \text{ for all } 1 \leq i \leq n\}. \quad (1)$$

Then the distance graph $G(\mathbb{Z}, D)$ has infinite chromatic number iff the exponent set Λ contains a strictly increasing sequence.

Future work. For a complete characterization of distance sets D having chromatic number finite, we need to drop or relax the condition of finite number of primes expressing the distance set D in Theorem 3. We believe that the characterization can be improved as follows.

Conjecture (K.–M.) Suppose that $D = \{d_1, d_2, \dots\}$. The chromatic number of Euclidean distance graph $G(\mathbb{Z}, D)$ is infinite iff some multiple of every integer appears in the set D and $\inf d_{i+1}/d_i = 1$.

To prove this conjecture concerning Euclidean distance graphs, the first step would be to prove the following conjecture that concerns p -adic distance graphs.

Conjecture (K.–M) A p -adic distance graph $G(\mathbb{Z}, D_p)$, with a p -adic distance set D_p , has infinite chromatic number iff there exist a set of finite distinct primes p_1, p_2, \dots, p_n and a set of n -tuples Λ over nonnegative integers that consists of a strictly increasing sequence such that D_p contains a set of distances that is the set of (1) written in terms of p -adic language.

5 $L(2, 1)$ -labeling for graphs

Extremal graph theory is, broadly speaking, the study of relations between various graph invariants, such as order, size, connectivity, minimum/maximum degree, chromatic number, etc., and the values of these invariants that ensure that the graph has certain properties. Since the first major result by Turan in 1941, numerous mathematicians have contributed to make this a vibrant and deep subject. Among all its topics, graph coloring is the most applicable and widely studied. Typically, a graph models conflicts, and a good coloring ensures partitions into parts with no conflicts. See Bollobás [7] and Jensen and Toft [28] for an overview.

Background. In ordinary graph coloring, adjacent vertices must be given different colors, and the actual values of the colors used are irrelevant. However, in many applications, it is also important to separate labels on vertices at farther distances, where the labels used have some numerical meaning. A natural problem of this type is the channel assignment problem, where channels (non-negative integers) are assigned to each radio transmitter (vertex) so that interfering (adjacent) transmitters get channels that are far apart. F.S. Roberts proposed a variation of the channel assignment problem, which Griggs and Yeh [25] introduced in 1992 and called the $L(2, 1)$ -labeling problem. Keeping the radio transmitter analogy in mind, vertices in a graph need to be labeled such that “close” vertices

(at distance 2) get different labels while “very close” vertices (at distance 1) get labels that are farther apart. More precisely, for a given graph G , a mapping $f : V(G) \rightarrow \mathbb{N} \cup \{0\}$ is called an $L(2, 1)$ -labeling if $|f(u) - f(v)| \geq 2$ for each edge uv of G and $|f(u) - f(v)| \geq 1$ for each pair $u, v \in V(G)$ at distance 2 apart. The $L(2, 1)$ -labeling number of G , denoted by $\lambda(G)$, is the smallest number t such that G has a $L(2, 1)$ -labeling that does not use any label greater than t .

As in the case of chromatic number of graphs, the maximum degree of a graph, $\Delta(G)$, is a natural candidate for bounding $\lambda(G)$. The obvious lower bound for λ is $\Delta + 1$, which holds with equality for the star $K_{1, \lambda}$. A greedy labeling (as shown in [25]) gives $\lambda(G) \leq \Delta^2 + 2\Delta$. This upper bound was improved to $\Delta^2 + \Delta$ in [9]. Griggs and Yeh [25] conjectured that for every graph G , $\lambda(G) \leq \Delta^2(G)$. This has been a motivating problem for research in this field, and some results are known. Note that it is enough to consider connected, regular graphs. Tight bounds have been obtained for special classes of graphs like paths, cycles, wheels, complete k -partite graphs and graphs with diameter 2 [25], trees [9, 25], etc. Some bounds have also been obtained for various other graph families like chordal graphs and unit interval graphs [48], hypercubes [25, 37, 54], and planar graphs [37]. See [6] for a wide ranging survey including algorithms, complexity and applications to communication networks. However, the core of the conjecture remains wide open – even for 3-regular graphs.

Results and future work. I proved the Griggs–Yeh Conjecture for 3-regular Hamiltonian graphs [31]. The proof is rather intricate, and requires the study of structural properties of the involved graphs. It starts by pre-labeling G to produce a graph H of ‘badly’ labeled pairs of vertices, and then it uses the structure of H to reduce finding $L(2, 1)$ -labeling of G to finding an ordinary coloring of H satisfying some additional constraints.

I also studied $L(2, 1)$ -labeling of two special graphs, the incidence graph of the projective plane of order q , and the Kneser graph. These graphs are very interesting and frequently occur in a variety of problems. Since they have a rich structure and it is not trivial to analyze most graph parameters for them, doing so can lead to insights into the more general problem.

The Kneser graph $K(m, k)$ is the disjointness graph on the k -subsets of $\{1, 2, \dots, m\}$. For $K(2k+1, k)$, in [32] I showed that $\lambda(G) \leq 4k + 2$. Here, the $L(2, 1)$ -labeling is obtained from a classification of structures between and within the color classes of a special vertex coloring.

For the incidence graph G of the projective plane $PG(2, q)$, Füredi and I [22] show that $\lambda(G) = q^2 + q = \Delta^2 - \Delta$. (The problem was also studied in [37]) To prove this result, we considered packing bipartite graphs into a complete bipartite graph and proved a sufficient condition for such a packing, which is analogous to the result of Sauer and Spencer [49] for packing graphs into a complete graph. For given bipartite graphs G_1 and G_2 with bipartitions X_1, Y_1 and X_2, Y_2 , respectively, a *packing of G_1 and G_2 into $K_{m, n}$* maps $X_1 \rightarrow [m], Y_1 \rightarrow [n]$ and $X_2 \rightarrow [m], Y_2 \rightarrow [n]$ injectively such that $E(G_1) \cap E(G_2) = \emptyset$. We showed that $2\Delta(G_1)\Delta(G_2) < 1 + \max\{m, n\}$ is a sufficient condition for such a packing.

The result on 3-regular Hamiltonian graphs is the first significant progress towards the Griggs and Yeh conjecture in the last few years. However, the extra condition of Hamiltonicity needs to be removed (to complete the proof for 3-regular graphs). A Hamiltonian cycle can be thought as a 2-factor consisting of one cycle. I am currently working on this with D. West by considering 3-regular graphs with 2-factors consisting of arbitrarily many cycles. Note that every 2-edge-connected 3 regular graph has a 2-factor.

We are also working on extending the ideas from the incidence graph of $PG(2, q)$ to a more general class of bipartite graphs. Füredi and I have succeeded in classifying the case $\Delta = 3$ and are pursuing other cases.

6 Rectilinear equilateral sets in \mathbb{R}^n

Background. This is a 20-year-old problem in combinatorial geometry. Kusner [26] conjectured that the maximum size of a set whose elements are pairwise equidistant under the ℓ_1 -norm in \mathbb{R}^n is $2n$. If it is true, this would be sharp. The conjecture has been proved for $n = 3$ [3] and $n = 4$ [38]. Recently, Alon and Pudlak [1] gave an upper bound $O(n \ln n)$. These proofs are very involved and use ideas from embeddings of metric spaces, approximation theory, etc. However, in the words of the authors, their methods won't solve the full conjecture.

Results and future work. Since this problem has the same flavor as that of computing $\chi(\mathbb{R}_1^n, 1)$, I have worked on it as well. I have made some progress on it.

I have converted the problem into a certain coloring of the sphere. I have showed that the Kusner conjecture is true if there exists an equidistant set of size $3n/2$ on the surface of the generalized-octahedron with radius 1 in \mathbb{R}^n . Related to this, I have showed that the maximum size of an equidistant set on the surface of the unit generalized-octahedron without its extreme point is at most n , and this is sharp.

I have also related this problem to the piercing (transversal) number of convex bodies in \mathbb{R}^n . I have shown that if \mathcal{F} is a family of the unit generalized-octahedrons such that every two members of \mathcal{F} intersect, then every three members of \mathcal{F} must intersect. I am working on extending this result to showing that a same \mathcal{F} with at least $2n + 1$ members must be an intersecting family. This would be enough to prove the Kusner conjecture.

7 The persistence of a number

Background. In the sequence 679, 368, 168, 48, 32, 6, each term is the product of the decimal digits of the previous one. Neil Sloane [50] defines the *persistence* of a number as the number of steps (five in the example) before the number collapses to a single digit. The numbers with persistence 1, 2, \dots , 11 have been found, and there is no number less than 10^{50} with persistence greater than 11. Sloane conjectured that there is a number d such that no number has persistence greater than d .

In the binary representation, the maximum persistence is 1. In the ternary representation, the second term is zero or a power of 2. It is conjectured that all powers of 2 greater than 2^{15} contain a zero when written in ternary. (The number 2^{15} consists of only 1s and 2s in ternary.) This is true up to 2^{500} . The importance of this conjecture is that the maximum persistence in the ternary representation is 3.

Sloane's general conjecture is that, for a given positive integer b , there is a number $d(b)$ such that the persistence in base b (that is, the b -ary representation) is at most $d(b)$.

Results and future work. I have worked on the conjecture of the ternary representation. Observe that if 2^n contains 12 in its ternary representation, 2^{n+1} must contain zero. I have proved that the power of 2 contains 0 or 12 in ternary. This implies that at least half of the powers of 2 contain zero in their ternary representations. This partial result is proved by considering only special ternary representations – consecutive 1s and 2s. I am working on extending the method to arbitrary appearances of 1s and 2s to complete the conjecture.

8 Security number

Background. A *defensive alliance* in a graph $G = (V, E)$ is a subset S of V with the property that every vertex $x \in S$ has as many neighbors, including x itself, in S as those in $V \setminus S$ [39]. The various concepts of alliances in graphs are motivated from a security issue that whether defenders

of x in S can defeat the attackers of x in $V \setminus S$. Recently, Brigham, Dutton, and Hedetniemi [8] introduced a global concept of an alliance: A subset $S \subset V$ is a *secure* set if every subset $U \subset S$ has as many neighbors, including U , in S as those in $V \setminus S$. We seek the parameter $\rho(G)$, the *security number* of G , that is the minimum size of a secure set in G .

It is easy to construct a graph having $\rho(G)$ arbitrarily small compared to $n(G)$, the number of vertices of G , in which the connectivity of the graph is low. Hence we are interested in how large $\rho(G)$ can be in terms of $n(G)$, and consider graphs with high connectivity. For instance, a complete graph of order n has the security number $\lceil n/2 \rceil$. Brigham, Dutton, and Hedetniemi [8] asked whether $\lceil n/2 \rceil$ is an upper bound for $\rho(G)$ for every graph G . In particular, Brigham (personal communication) asked whether the security number of Kneser graphs $K(m, k)$ is at most $\lceil n/2 \rceil$ where $n = \binom{m}{k}$.

Results and future work. I have proved that $\rho(K(m, 2)) = \lceil \frac{n+1}{2} \rceil$ (independently by Dutton–Lee–Brigham [10]). This is a negative answer to their question and makes the problem more interesting. In my proof of $\rho(K(m, 2)) = \lceil \frac{n+1}{2} \rceil$, I use the language of matching and covering numbers. I am working on extending this idea to obtain the security number for the whole class of Kneser graphs.

9 Additional topics for current and future work

In this section, I will describe some of the specific topics that I am working on or intend to work on in the near future. In general, I am interested in problems from a variety of topics such as extremal graph theory, set systems, and combinatorial geometry, among others.

Graph coloring extensions As mentioned earlier, graph coloring is highly applicable, and at times these applications have motivated some extensions of the notion of graph coloring. I have already worked on $L(2, 1)$ -labeling. Another such interesting problem is strong edge-coloring, an edge coloring in which edges at distance 1 or 2 receive distinct colors. In 1985, Erdős and Nešetřil conjectured that $s'(G)$, smallest number of colors needed in such a coloring, is at most $(5/4)\Delta^2(G)$. Faudree, et al. [16] conjectured that $s'(G) \leq \Delta^2(G)$ when G is bipartite. Some partial progress has been made on these and other such conjectures. I intend to work on this topic and, as a start I showed that $s'(K(2k+1, k)) = 2k+1$ in [30].

A related generalization is $\chi(G^2)$, where G^2 is the graph on $V(G)$ with two vertices adjacent if they are at distance 1 or 2 in G . Strong edge-coloring can be thought of as the study of $\chi(G^2)$ for line graphs. It is also related to $L(2, 1)$ -labeling. Since $\Delta^2(G)$ is also an upper bound on $\Delta(G^2)$, I expect $\chi(G^2)$ to be a good lower bound on $\lambda(G)$. This makes the study of $\chi(G^2)$ worthwhile. For instance, finding a lower bound on $\chi(K(2k+1, k)^2)$ or its easier variation, an upper bound on the independence number of $K(2k+1, k)^2$, are both challenging questions that could lead to non-trivial lower bounds on $\lambda(K(2k+1, k))$ complementing the upper bound mentioned earlier.

Euclidean Ramsey Theory. Ramsey theory typically deals with problems of the following type. Given a set S , a family \mathcal{F} of subsets of S , and $r \in \mathbb{Z}^+$, is it true that in every partition of $S = C_1 \cup \dots \cup C_r$ into r subsets, there is some C_i that contains some $F \in \mathcal{F}$. In Euclidean Ramsey theory, S is taken to be some Euclidean space, and the sets in \mathcal{F} are determined by various geometric configurations. This field was started by a series of papers by Erdős, Graham, Montgomery, Spencer, etc. (see [24] for a survey).

Recalling its definition, metric space coloring turns out to be a special type of Euclidean Ramsey problem. Due to this connection, I have become interested in this area. This field provides a common framework for a variety of topics from geometry and number theory, and it has a lot of interesting

open problems that I intend to work on.

The Kneser graph $K(m, k)$. As mentioned in Section 5, the Kneser graph is an interesting graph to explore for its own sake, especially for its relationship to many topics. For instance, it can be found in connection with combinatorial geometry in the papers of Lovasz [41] and Bárány [4], in connection with extremal combinatorics in Füredi–Griggs–Kleitman [19], intersecting families in Erdős–Ko–Rado [13], Frankl–Füredi [17], etc. In my research, I have studied its $L(2, 1)$ -labeling, strong edge-coloring, and security number, and I also believe that it has relevance to finding good lower bounds on $\chi(\mathbb{Z}_1^n, r)$. I am also interested in the conjecture that the distinguishing number (least number of labels needed to break all symmetries) of Kneser graph $K(m, k)$ is 2, for m large enough. So, I would like to further study problems about the Kneser graph and explore its connections with other problems.

Intersecting families. Bounding the size of a family whose members satisfy certain intersection properties is one of the most fundamental problems in combinatorics of finite sets. Such results are widely applicable and provide good extremal constructions (see Babai and Frankl [2] for numerous such results). We used the Frankl–Wilson result on intersecting families to construct a lower bound on $\chi(\mathbb{R}_p^n, 1)$. A better or different such result might be needed to improve the lower bounds on $\chi(\mathbb{R}_p^n, 1)$. Also, for $L(2, 1)$ -labeling, a lower bound on $\chi(K(2k + 1, k)^2)$ could be reformulated as “what is the maximum size of a k -uniform family \mathcal{F} of subsets of a $2k + 1$ -element set, satisfying the condition $1 \leq |F \cap F'| \leq k - 1$ for all $F, F' \in \mathcal{F}$?”. Such an \mathcal{F} is an independent set in $(K(2k + 1, k))^2$.

Packing and covering of triangles. Let $\nu(G)$ be the maximum size of a set of pairwise edge-disjoint triangles of G , and let $\tau(G)$ be the minimum number of edges that cover all the triangles of G . We want to bound $\tau(G)$ in terms of $\nu(G)$. Note that $\tau(G)$ has trivial lower bound $\nu(G)$ and upper bound $3\nu(G)$. The graphs $G = K_4$ and K_5 show that $\tau(G)$ can be as large as $2\nu(G)$. In 1981, Zs. Tuza [51] conjectured that $\tau(G) \leq 2\nu(G)$, which has drawn a great deal of attention. Despite some progress, the conjecture still remains open. I am interested in such packing and covering problems on graphs and hypergraphs.

References

- [1] N. ALON, P. PUDLAK: Equilateral sets in l_p^n . *Geometric and Functional Analysis*, **13** (2003), 467–482.
- [2] L. BABAI, P. FRANKL: *Linear Algebra Methods in Combinatorics, with Applications to Geometry and Computer Science*, Preliminary version 2, University of Chicago, 1992.
- [3] H.J. BANDELDT, V. CHEPOI, M. LAURENT: Embedding into rectilinear spaces. *Discrete Comp. Geom.* **19** (1998), 595–604.
- [4] I. BÁRÁNY: A short proof of Kneser’s conjecture. *J. Combin. Theory Ser. A.* **25** (1978), 325–326.
- [5] M. BENDA, M.A. PERLES: Coloring of metric spaces. *Unpublished*.
- [6] H.L. BODLAENDER, T. KLOKS, R.B. TAN, J. VAN LEEUWEN: λ -coloring of graphs. Preprint. (Preliminary version in *Lecture Notes in Comput. Sci.*, **1770**, Springer, Berlin, 2000, 395–406.)
- [7] B. BOLLOBAS: *Extremal Graph Theory*. Academic Press, London, 1978.
- [8] R. BRIGHAM, R. DUTTON, S. HEDETNIEMI: Security in graphs. Preprint, 2005.
- [9] G.J. CHANG, D. KUO: The $L(2, 1)$ -labeling problem on graphs. *SIAM J. Disc. Math.* **9** (1996), 309–316.
- [10] R. DUTTON, R. LEE, R. BRIGHAM: Bounds on a graph’s security number. Preprint, 2005.
- [11] R. B. EGGLETON, P. ERDŐS, D. K. SKILTON: Coloring the real line, *J. Combin. Theory B* **39** (1985), 86100.
- [12] R. B. EGGLETON, P. ERDS, D. K. SKILTON: Colouring prime distance graphs. *Graphs Combin.* **6** (1990), no. 1, 17–32.
- [13] P. ERDŐS, CHAO KO, R. RADO: Intersections theorems for systems of finite sets. *Quart. J. Math. Oxford* **12** (1961), 313–320
- [14] P. ERDŐS, C. ROGERS: Covering space with convex bodies. *Acta Arithmetica* **7** (1962), 281–285.

- [15] M. FERRARA, Y. KOHAYAKAWA, V. RÖDL: Distance graphs on the integers. *Combin. Probab. Comput.* **14** (2005), no. 1-2, 107–131.
- [16] R.J. FAUDREE, A. GYARFAS, R.H. SCHELP, ZS. TUZA: Induced matchings in bipartite graphs. *Disc. Math.* **78** (1989), no. 1-2, 83–87.
- [17] P. FRANKL, Z. FÜREDI: Extremal problems concerning Kneser graphs. *J. Combin. Theory Ser. B* **40** (1986), 270–284.
- [18] P. FRANKL, R.M. WILSON: Intersection theorems with geometric consequences. *Combinatorica* **1** (1981), 357–368.
- [19] Z. FÜREDI, J.R. GRIGGS, D.J. KLEITMAN: Pair labellings with given distance, *SIAM J. Disc. Math.* **2** (1989), 491–499.
- [20] Z. FÜREDI, J-H. KANG: Distance graph on \mathbb{Z}^n with ℓ_1 -norm. *Theoret. Comput. Sci., Special issue on Combinatorics of the Discrete Plane and Tilings.* **319** (2004), 357–366.
- [21] Z. FÜREDI, J-H. KANG: Covering n -space by convex bodies and its chromatic number. *Disc. Math., Special issue in honor of M. Simonovits*, accepted for publication.
- [22] Z. FÜREDI, J-H. KANG: $L(2, 1)$ -labelling and packing of bipartite graphs, preprint.
- [23] F. Q. GOUVEA: *p -adic Numbers. An Introduction*, Universitext. Springer-Verlag, Berlin, 1993.
- [24] R. GRAHAM : Euclidean Ramsey theory, *Handbook of Discrete and Computational Geometry*. CRC Press (1997), 153–166.
- [25] J.R. GRIGGS, R.K. YEH: Labelling graphs with a condition at distance 2. *SIAM J. Disc. Math.* **5** (1992), 586–595.
- [26] R. GUY, editor, Unsolved Problems: An Olla-Podria of Open Problems, Often Oddly Posed. *Amer. Math. Monthly.* **90** (1983), 196–200.
- [27] R. GUY: *Unsolved Problems in Number Theory*. Springer, 3 edition (2004).
- [28] T. JENSEN, B. TOFT: *Graph Coloring Problems*. John Wiley & Sons, Inc., (1995).
- [29] K. JONAS: Graph colorings analogues with a condition at distance two: $L(2,1)$ -labelling and list λ -labellings, Ph.D. thesis, University of South Carolina (1993).
- [30] J-H. KANG: Covering and coloring of metric spaces, and $L(2, 1)$ -labeling of graphs. Ph.D. thesis, University of Illinois at Urbana-Champaign (2004).
- [31] J-H. KANG: $L(2, 1)$ -labelling for Hamiltonian graphs of maximum degree 3, submitted for publication.
- [32] J-H. KANG: $L(2, 1)$ -labeling for Kneser Graph, preprint.
- [33] J-H. KANG: Security Number of Kneser graphs, preprint.
- [34] J-H. KANG, H. MAHARAJI: Distance graph in p -adic norm. *preprint*
- [35] N. KOBLITZ: *p -adic Numbers, p -adic Analysis, and Zeta-functions*, Graduate Texts in Mathematics, Vol. 58. Springer-Verlag, New York-Heidelberg, 1977.
- [36] H. HADWIGER: Überdeckungssätze für den Euklidischen Raum. *Portualiae Math.* **4** (1944), 140–144.
- [37] F. JOHN: Extremum problems with inequalities as subsidiary conditions. *Courant anniversary Volume*. New York, 1948, pp.187–204.
- [38] J. KOOLEN, M. LAURENT, A. SCHRIJVER: Equilateral dimension of the rectilinear space. *Des. Codes Cryptogr.* **21** (2000), 149–164.
- [39] P. KRISTIANSEN, S.M. HEDETNIEMI, S.T. HEDETNIEMI: Alliances in graphs. *JCMCC* **48** (2004), 157–177.
- [40] D. G. LARMAN, C. A. ROGERS: The realization of distances within sets in Euclidean space. *Mathematika* **19** (1972), 1–24.
- [41] L. LOVÁSZ: Kneser’s conjecture. *J. Combin. Theory Ser. A.* **25** (1978), 319–324.
- [42] J. MATOUSEK: *Lectures on discrete geometry*. Springer, New York, 2002.
- [43] M. MORAYNE: The chromatic number of the plane. *Presentation at MIGHTY Conference* (2002).
- [44] J. PACH : Finite point configurations, *Handbook of Discrete and Computational Geometry* CRC Press (1997), 3–18.
- [45] J. PACH, P. AGARWAL: *Combinatorial geometry*. John Wiley & Sons, Inc., New York 1995.
- [46] C. ROGERS: A note on coverings. *Mathematika* **4** (1957), 1–6.
- [47] I. Z. RUZSA, ZS. TUZA, M. VOIGT: Distance graphs with finite chromatic number. *J. Combin. Theory Ser. B* **85** (2002), no. 1, 181–187.
- [48] D. SAKAI: Labelling chordal graphs: distance two condition. *SIAM J. Disc. Math.* **7** (1994), 133–140.
- [49] N. SAUER, J. SPENCER: Edge disjoint placement of graphs. *J. Combin. Theory Ser. B* **25** (1978), no.3, 295–302.
- [50] N. SLOANE: The persistence of a number. *J. Recreational Math.* **6** (1973), 97–98.
- [51] ZS. TUZA: Conjecture, Finite and Infinite Sets. Eger, Hungary, 1891, A. Hajnal, L. Lovász, V.T. Sós (Eds.), *Proc. Colloq. Math. Soc. J. Bolayí*, **37**, North-Holland, Amsterdam, 1984, p.888.
- [52] M. VOIGT: On the chromatic number of distance graphs. *J. Inform. Process. Cybernet. EIK* **28** (1992), 2128.

- [53] M. VOIGT, H. WALTHER: Chromatic number of prime distance graphs. *Disc. Applied Math.* **51** (1994), 197-209.
- [54] M.A. WHITTLESEY, J.P. GEORGES, D.W. MAURO: On the λ -number of Q_n and related graphs. *SIAM J. Disc. Math.* **8** (1995), 499–506.
- [55] D.R. WOODALL: Distances realized by sets covering the plane. *J. Combinatorial Theory Ser. A* **14** (1973), 187–200.
- [56] V. YEGNANARAYANAN On a question concerning prime distance graphs. *Discrete Math.* 245 (2002), no. 1-3, 293–298.
- [57] XUDING ZHU: Pattern periodic coloring of distance graphs. *J. Combin. Theory Ser. B* 73 (1998), no. 2, 195–206.