

Research Statement

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1 Introduction

My research interests lie in Discrete Mathematics, especially Combinatorics, Graph Theory, Combinatorial Geometry, and Combinatorial Number Theory. For me, the most exciting aspect of working in discrete mathematics is the prevalence of combinatorial problems in various fields of mathematics and various applications to Computer Science and real life problems such as building transmitters in a town, assignment of radio channels, scheduling meetings, etc. Graph theory and combinatorics provide a unifying abstract structure for these problems. At the same time, significant progress in combinatorial problems requires application of tools/methods from a variety of fields like probability, geometry, algebra, number theory, etc. I have experienced both these aspects of research in combinatorics through my work on geometric combinatorial problems and number theoretic combinatorial problems: my proofs employ ideas from discrete Geometry, probability, asymptotic analysis, and number theory. [21, 22, 36]. In the long term, I will continue to deepen and broaden my understanding of fields like geometry, algebra, number theory, and probability, as well as combinatorics, so that I can make contributions across a wide range of topics which lie in the intersection of these fields with combinatorics.

In this statement, I will discuss my research and results to date and then briefly describe other topics that I have worked on or intend to in the near future.

2 Coloring metric spaces

Combinatorial geometry is the study of combinatorial properties of fundamental geometric objects, whose origins go back to antiquity. It has come into maturity in the last century through the seminal works of O. Helly, K. Borsuk, P. Erdős, H. Hadwiger, L. Fejes Tóth, B. Grünbaum and many other excellent mathematicians who initiated new combinatorial approaches to classical questions studied by Newton, Gauss, Minkowski, and Hilbert, as well as new areas of investigation. The textbooks by Matousek [45] and, Pach and Agarwal [48] provide an overview of the topics and methods.

Background. A fundamental approach to studying the combinatorial structure of a metric space is: If we partition a space into a small number of parts (i.e., color its points), at least one of these parts must contain certain “unavoidable” point configurations. In the most basic case, the configuration consists of a pair of points at a given distance (which can usually be scaled to one). The first such question, known as the Hadwiger–Nelson problem, was: What is the minimum number of colors needed for coloring the plane so that no two points at unit distance receive the same color? The answer is known to be between 4 and 7, with no improvement in the last 50 years. The same question can be asked for any metric space in high dimensions. This has turned out to be a challenging problem with very little progress made on it. See Pach [47] for a short survey. We consider ℓ_p -spaces and investigate $\chi(\mathbb{R}_p^n, 1)$, i.e., the chromatic number of the unit distance graph on \mathbb{R}^n under the ℓ_p -norm. Almost nothing is known about these chromatic numbers, with the best results for the Euclidean space ($p = 2$), $(1.2 + o(1))^n \leq \chi(\mathbb{R}_2^n, 1) \leq (3 + o(1))^n$. The lower bound was given by Frankl and Wilson [19], and the upper bound by Larman and Rogers [42] in 1981 and 1972, respectively. Other than this, only weak bounds on $\chi(\mathbb{R}_2^3, 1)$ are known. The subgraph induced by rational points has also been investigated in small dimensions: $\chi(\mathbb{Q}_2^2, 1) = 2$ (Woodall [59]), $\chi(\mathbb{Q}_2^3, 1) = 2$ and $\chi(\mathbb{Q}_2^4, 1) = 4$

(Benda and Perles [8]) are the main results. Morayne [46] used Fermat’s Last Theorem to observe that $\chi(\mathbb{Q}_p^2, 1) = 2$ for $p \geq 3$.

Results. In [21, 22, 32], Füredi and I investigate $\chi(\mathbb{R}_p^n, 1)$ for general $p \geq 1$, by studying certain fundamental geometric ideas, namely, partitions of \mathbb{R}^n , sphere packing in \mathbb{R}^n , and covering of \mathbb{R}^n by translates of a convex body, which respectively led to the upper bounds, $\sqrt{p/(2\pi n)}(5(ep)^{1/p})^n$ (for all n), 9^n (for all n), and $c(n \ln n)5^n$ (for large n). The third upper bound requires a covering with special characteristics, described in the following section. We use an intersection inequality of Frankl and Wilson [19] to construct a finite subgraph of $(\mathbb{R}_p^n, 1)$ with small independence number and, consequently, large chromatic number, which gives a lower bound of $(1.139)^n$ for all p .

Future work. To improve the upper bounds, the first step could be to combine the packing and covering arguments used in [32] and [22]. A fundamental improvement might depend on a better understanding of the coloring of the $(n - 1)$ -dimensional sphere of radius a as a subgraph of \mathbb{R}^n . Except for a lower bound of n by Lovász (see [47]) and some bounds for small dimensions, almost nothing is known; we plan to further investigate the problem.

The exponential lower bound follows from an application of the Frankl-Wilson Inequality to a carefully defined subset of $\{0, 1\}^n$ under scaling. We try to improve the lower bound, possibly by considering a bigger subspace like $\{-1, 0, 1\}^n$.

3 Covering Euclidean n -space

Roughly speaking, the *density* of a covering of \mathbb{R}^n is the average number of bodies needed to cover a given unit space. The density of a cover measures how good the cover is. We seek to minimize the density of a collection \mathcal{C} of convex bodies that covers \mathbb{R}^n .

Rogers [49] proved that, for a given closed convex body C in the n -dimensional Euclidean space, where $n \geq 3$, there is a covering for \mathbb{R}^n by translates of C with density $O(n \ln n)$. However, low density does not imply low multiplicity of the covering, where *multiplicity* is the number of copies of $C \in \mathcal{C}$ containing each point. Even though the global density of a covering is low, there can exist local clusters of high multiplicity. Later, Erdős and Rogers [15] showed that, for sufficiently large n , there exists such a covering with not only density at most $O(n \ln n)$ but also multiplicity at most $O(n \ln n)$. In [22], Füredi and I give a new combinatorial proof of this covering result using methods and tools from probabilistic combinatorics and discrete geometry, and asymptotic analysis, such as Lovasz Local Lemma and a deep theorem on volume ratio by K. Ball. We use this special covering in the proof of the upper bound $c(n \ln n)5^n$ on $\chi(\mathbb{R}_p^n, 1)$.

4 Distance graph in p -adic norm

While a coloring of a metric space in high dimensions has a flavor of Combinatorial Geometry, an analogous question asked for the integer line has more of a flavor of Combinatorial Number Theory. We want to partition the integer line so that any part avoids a pair whose difference belong to a prescribed, so-called, *distance set* of positive integers.

Background. The integer *distance graph* $G(\mathbb{Z}, D)$ with distance set $D = \{d_1, d_2, \dots\}$ has the set of integers \mathbb{Z} as the vertex set and two vertices $x, y \in \mathbb{Z}$ are adjacent if and only if $|x - y| \in D$. The integer distance graphs (under Euclidean norm) were first systematically studied by Eggleton–Erdős–Skilton in 1985 [12, 13], and have been investigated in many ways [50, 56, 57, 61]. One of main goals in these problems is characterizing prescribed distance sets that make the corresponding distance

graphs to have finite chromatic number. Ruzsa, Tuza, and Voigt [50] gave a sufficient condition for $\chi(G(\mathbb{Z}, D))$ to be finite:

Theorem 1 (Ruzsa–Tuza–Voigt [50]) *If $\inf d_{i+1}/d_i > 1$, then $\chi(G(\mathbb{Z}, D))$ is finite.*

Hence it is needed to investigate distance sets D having $\inf d_{i+1}/d_i = 1$ for the characterization of distance sets D with finite chromatic number.

Results. Maharaj and I approaches the problem from the viewpoint of p -adic norms in [36]. For a pair of integers x and y , we write $x|y$ if x divides y , and $x \nmid y$ if x doesn't divide y . Let p be a prime number. Then any rational number x can be uniquely written in the form $x = \frac{r}{s}p^\ell$ where $\ell \in \mathbb{Z}$ and r, s are integers not divisible by p . One defines the p -adic norm of x by $\|x\|_p := 1/p^\ell$. This gives rise to a non-Archimedean norm on the rationals \mathbb{Q} .

The p -adic norm not only itself is an important norm in number theory but also has a natural interpretation into Euclidean norm that allows us to nicely describe the class of distance sets D with $\inf d_{i+1}/d_i = 1$. More precisely, for an integer $x > 1$, the interpretation of p -adic distance into Euclidean distance (and vice versa) is done through the product formula $|x| \prod_p \|x\|_p = 1$, where the product is over all prime numbers p , and $|\cdot|$ is the Euclidean norm. In [36], Maharaj and I have given bounds on the chromatic number, including several exact ones, that depend on the divisibility properties of the numbers d_i and that are applicable even in the case that $\inf d_{i+1}/d_i = 1$.

For example, (since Euclidean distance graphs have been extensively studied, we state here in terms of the usual Euclidean norm) suppose that p_1, p_2, \dots, p_n are distinct prime numbers and D_i is a finite set of distinct non-negative powers of p_i of size $k_i := |D_i|$ for each $i = 1, 2, \dots, n$. If $D := \{aq : \text{For some } 1 \leq i \leq n, q \in D_i \text{ in which case } p_i \nmid a\}$, then $\chi(G(\mathbb{Z}, D)) = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$. If $D := \{aq_1 q_2 \dots q_t : \text{For each } 1 \leq i \leq n, q_i \in D_i \text{ and } p_i \nmid a\}$, then $\chi(G(\mathbb{Z}, D)) = \min(p_i^{k_i} : 1 \leq i \leq n)$.

One of our main results states a sufficient condition for an Euclidean distance graph $G(\mathbb{Z}, D)$ to have finite chromatic number as follows.

Theorem 2 (K.–M. [36]) *Let $D = \{d_1, d_2, \dots\}$ be a given distance set. For each prime number p , let $D(p)$ be the set of all powers p^n of p such that p^n divides d_i but p^{n+1} does not divide d_i for some i . Then*

$$\chi(G(\mathbb{Z}, D)) \leq \min(p^{|D(p)|} : p \text{ is prime}).$$

For example, if D is any set of odd numbers, then $\chi(G(\mathbb{Z}, D)) \leq 2$ since $D(2) = \{1\}$. Observe that it follows from Theorem 2 that if a distance graph $G(\mathbb{Z}, D)$ has infinite chromatic number, then arbitrarily high powers of every prime number appear as divisors of the numbers in the distance set D . Theorem 2 can be viewed as complementing the Theorem 1 of Ruzsa–Tuza–Voigt. For example, let $p_1 < p_2 < \dots$ be an enumeration of the prime numbers. Set $D = \{d_1, d_2, \dots\}$ where $d_i := (p_1 p_2 \dots p_i)^i$ for each i . Then by Theorem 1, $\chi(G(\mathbb{Z}, D))$ is finite but Theorem 2 is inconclusive. On the other hand, if D is the set of all positive integers not divisible by a fixed prime number p (so $D(p) = \{1\}$), then Theorem 2 implies that $\chi(G(\mathbb{Z}, D)) \leq p$ while Theorem 1 is inconclusive. In this sense, Theorems 1 and 2 complement each other.

Finally, we state our current strongest result as Theorem 3, which gives a conditional characterization of a distance set having finite chromatic number. Let Λ be a subset of n -tuples over nonnegative integers \mathbb{N}_0^n . We define an order $(e_1, e_2, \dots, e_n) < (e'_1, e'_2, \dots, e'_n)$ in Λ if $e_i < e'_i$ for each $1 \leq i \leq n$.

Theorem 3 (K.–M. [36]) *Let p_1, p_2, \dots, p_n be distinct prime numbers. Let $\Lambda \subset \mathbb{N}_0^n$. Define*

$$D := \{ap_1^{e_1} p_2^{e_2} \dots p_n^{e_n} : (e_1, e_2, \dots, e_n) \in \Lambda, a \in \mathbb{Z} \text{ with } p_i \nmid a \text{ for all } 1 \leq i \leq n\}. \quad (1)$$

Then the distance graph $G(\mathbb{Z}, D)$ has infinite chromatic number iff the exponent set Λ contains a strictly increasing sequence.

Future work. For a complete characterization of distance sets D having chromatic number finite, we need to drop or relax the condition of finite number of primes expressing the distance set D in Theorem 3. We believe that the characterization can be improved as follows.

Conjecture 1. (K.–M.) Suppose that D is a given distance set. The chromatic number of Euclidean distance graph $G(\mathbb{Z}, D)$ is infinite iff for every finite partition of $D = \cup_{1 \leq j \leq k} D_j$, there exists j , $1 \leq j \leq k$, such that some multiples of every integer appear in the set $D_j = \{d_1 < d_2 < \dots\}$ and $\inf_{d_i \in D_j} d_{i+1}/d_i = 1$.

To have Conjecture 1 that concerns Euclidean distance graphs, the first step would be proving the following conjecture that concerns p -adic distance graphs.

Conjecture 2. (K.–M) A p -adic distance graph $G(\mathbb{Z}, D_p)$, with a p -adic distance set D_p , has infinite chromatic number iff there exist a set of finite distinct primes p_1, p_2, \dots, p_n and a set of n -tuples Λ over nonnegative integers that consists of a strictly increasing sequence such that D_p contains a set of distances that is the set of (1) written in terms of p -adic language.

Another natural generalization of the distance graph problem on \mathbb{Z} is as follows. We use a specific example to illustrate the idea. Let $K = \mathbb{Q}(i)$ where $i^2 = -1$ and let $R = \mathbb{Z}[i]$. Then R is the ring of integers of K , that is, R is the integral closure of \mathbb{Z} in K . As an Abelian group, R is isomorphic to \mathbb{Z}^2 and geometrically R is a lattice in the plane. A prime p of \mathbb{Z} gives rise to two p -adic norms on R if $p \equiv 1 \pmod{4}$, otherwise p gives rise to a single p -adic norm on R (see [4]). The product formula $|x| \prod_p \|x\|_p = 1$ still holds, so specifying an Euclidean norm is tantamount to specifying several p -adic norms. The same type of questions considered in this section can be asked about distance graphs on R .

In general, if R is a ring of integers in a number field K , then it is possible for R to have several inequivalent archimedean norms. An example of this would be the ring of integers of the field $\mathbb{Q}(\sqrt[3]{2})$ which has two complex embeddings and one real embedding so there are two inequivalent Euclidean norms. It would be interesting to study how the structure of the distance graphs would change when one transitions from \mathbb{Z} to general rings of integers. It is hoped that by studying such generalisations, light would be shed on the standard distance graph problem on \mathbb{Z} . Furthermore, it is known that if K is a degree n extension of \mathbb{Q} then the ring R is a free \mathbb{Z} -module of rank n . So studying distance graphs on R is tantamount to studying a class of distance graphs on the lattice \mathbb{Z}^n . In the next section we consider a class of distance graphs on \mathbb{Z}^n with respect to the ℓ_1 norm.

5 Distance Graphs on \mathbb{Z}^n

Background. One generalization Hadwiger-Nelson problem is the considering of the n -dimensional real space under ℓ_p -norm, which has geometric aspects. Another natural candidate is the discrete space \mathbb{Z}^n . In this case, the ℓ_1 -norm is the most natural metric. For a positive integer r , by (\mathbb{Z}^n, r) we mean the graph with vertex set \mathbb{Z}^n and two vertices x, y are adjacent iff $\|x - y\|_1 = r$ where $\|\cdot\|_1$ is the ℓ_1 norm throughout in this section. More generally, for $D \subseteq \mathbb{N}$, by (\mathbb{Z}^n, D) we mean the graph with vertex set \mathbb{Z}^n and two vertices x, y are adjacent iff $\|x - y\|_1 \in D$. For the case $D = \{r : 1 \leq r \leq d - 1\}$ where $d \leq n$, Louge[43] use results from coding theory to give various lower bounds of the chromatic number. Some of them are

Theorem 4 Let $G = (\mathbb{Z}^n, 1 \leq r \leq d - 1)$. Then

$$\chi(G) \geq \frac{2^n}{A(n, d)} \quad (2)$$

and

Theorem 5 *There exists a sequence of positive integers $n_1 < n_2 < \dots$ and d_1, d_2, \dots such that $d_i \leq n_i$ for all i and $\frac{d_i}{n_i} \rightarrow \delta$ with*

$$\lim_{i \rightarrow \infty} \frac{\log_2 \chi(G_i)}{n_i} \geq 1 - \alpha_2(\delta)$$

where $G_i = (\mathbb{Z}^{n_i}, 1 \leq r \leq d_i - 1)$.

Note that the subscript p of \mathbb{Q} indicates that the ℓ_p norm is being used.

Results. In [21], Füredi and I investigate (\mathbb{Z}^n, r) . Since the graph (\mathbb{Z}_1^n, r) is bipartite when r is odd, we study $\chi(\mathbb{Z}_1^n, r)$ for even r . We prove that $\chi(\mathbb{Z}_1^n, 2) = 2n$ for all n , but $\chi(\mathbb{Z}^n, r) \geq 2n + 1$ for $n \geq 3$ and even $r \geq 4$. In addition, we give an exponential lower bound $(1.139)^n$ by constructing an appropriate finite subgraph, again using the Frankl-Wilson inequality. Since (\mathbb{Z}_1^n, r) is a subgraph of (\mathbb{R}_1^n, r) , we have $\chi(\mathbb{Z}_1^n, r) \leq \chi(\mathbb{R}_1^n, r) = \chi(\mathbb{R}_1^n, 1)$, so the upper bounds for $\chi(\mathbb{R}_1^n, 1)$ in Section 2 hold here. For small values of r , we have a more useful upper bound, $3r^{n-2}$ for all n and all even $r \geq 4$, which follows from a recurrence that extends a coloring on a selected hyperplane to the whole space.

Work in progress. In [1], the case $D = \{t : 1 \leq r < d_1 \text{ or } d_2 < r \leq n\}$ where $d_1 \leq d_2$ has been investigated, and it is being typed now. This turns out to involve more interesting stuff from coding theory, namely codes in spherical caps in Euclidean space.

Future work. We want to determine the clique number in the induced subgraph on the subset $\{0, 1\}^n$. The clique number not only serves the lower bound of $\chi(\mathbb{Z}^n, r)$ for a fixed constant r but also is important in coding theory to obtain information on the size of optimal codes with specified minimum distances.

One of the main goals is to prove analogues of Theorems 4 and 5 for more general distance sets. From the proof of Theorem 4, these results depend only on the subgraph induced on $\{0, 1\}^n$. One would expect better results if one considers the induced subgraph on $V' := \{0, 1, 2, \dots, q\}^n$. However, the distance no longer coincides with the Hamming distance. We would have to consider the following problem: find upper bounds on the size of a set $C \subset V'$ such that $\|x - y\|_1 \in \{0, 1, \dots, d\}$. If $\gamma(q, d)$ is such an upper bound then immediately we get that for $1 \leq d \leq n$, $\chi(\mathbb{Z}^n, 1 \leq r \leq d) \geq q^n / \gamma(q, d)$. It is expected that the techniques used in the papers [2, 3] would need to be adapted. We also expect better asymptotic results than those in Theorem 5. In order to adapt the results to graphs (\mathbb{Z}^n, D) to more general distance sets D , it would again be necessary to adapt the coding theoretic results appropriately. For example, if D is a non-empty subset of $\{1, \dots, n - 1\}$ we would have to find good upper bounds on the size of a largest subset C of $\{0, 1\}^n$ with the property that $d(x, y) \in D$ for all distinct $x, y \in C$. If the set D contains integers greater than $n - 1$, then we would have to consider subsets \mathcal{C} of the set $\{0, 1, 2, \dots, q - 1\}^n$ where q is the smallest integer such that $(q - 1) \cdot n \geq \max D$ and find good upper bounds on a largest subset \mathcal{C} of $\{0, 1, 2, \dots, q - 1\}^n$ with the property that $\|x - y\|_1 \in D$ for all distinct $x, y \in \mathcal{C}$. In both cases it is expected that the techniques of the papers [2, 3] to be adapted to solve these problems.

6 $L(2, 1)$ -labeling for graphs

Extremal graph theory is, broadly speaking, the study of relations between various graph invariants, such as order, size, connectivity, minimum/maximum degree, chromatic number, etc., and the values of these invariants that ensure that the graph has certain properties. Since the first major result by Turan in 1941, numerous mathematicians have contributed to make this a vibrant and deep subject. Among all its topics, graph coloring is the most applicable and widely studied. Typically,

a graph models conflicts, and a good coloring ensures partitions into parts with no conflicts. See Bollobás [10] and Jensen and Toft [30] for an overview.

Background. In ordinary graph coloring, adjacent vertices must be given different colors, and the actual values of the colors used are irrelevant. However, in many applications, it is also important to separate labels on vertices at farther distances, where the labels used have some numerical meaning. A natural problem of this type is the channel assignment problem, where channels (non-negative integers) are assigned to each radio transmitter (vertex) so that interfering (adjacent) transmitters get channels that are far apart. F.S. Roberts proposed a variation of the channel assignment problem, which Griggs and Yeh [27] introduced in 1992 and called the $L(2,1)$ -labeling problem. Keeping the radio transmitter analogy in mind, vertices in a graph need to be labeled such that “close” vertices (at distance 2) get different labels while “very close” vertices (at distance 1) get labels that are farther apart. More precisely, for a given graph G , a mapping $f : V(G) \rightarrow \mathbb{N} \cup \{0\}$ is called an $L(2,1)$ -labeling if $|f(u) - f(v)| \geq 2$ for each edge uv of G and $|f(u) - f(v)| \geq 1$ for each pair $u, v \in V(G)$ at distance 2 apart. The $L(2,1)$ -labeling number of G , denoted by $\lambda(G)$, is the smallest number t such that G has a $L(2,1)$ -labeling that does not use any label greater than t .

As in the case of chromatic number of graphs, the maximum degree of a graph, $\Delta(G)$, is a natural candidate for bounding $\lambda(G)$. The obvious lower bound for λ is $\Delta + 1$, which holds with equality for the star $K_{1,\lambda}$. A greedy labeling (as shown in [27]) gives $\lambda(G) \leq \Delta^2 + 2\Delta$. This upper bound was improved to $\Delta^2 + \Delta$ in [11], $\Delta^2(G) + \Delta(G) - 1$ by Kral and Skrekovski [38], and $\Delta^2(G) + \Delta(G) - 2$ by Goncalves [24]. Griggs and Yeh [27] conjectured that for every graph G , $\lambda(G) \leq \Delta^2(G)$. This has been a motivating problem for research in this field, and some results are known. Note that it is enough to consider connected, regular graphs. Tight bounds have been obtained for special classes of graphs like paths, cycles, wheels, complete k -partite graphs and graphs with diameter 2 [27], trees [11, 27], etc. Some bounds have also been obtained for various other graph families like chordal graphs and unit interval graphs [51], hypercubes [27, 40, 58], and planar graphs [40]. See [9] for a wide ranging survey including algorithms, complexity and applications to communication networks. However, the core of the conjecture remains wide open – even for 3-regular graphs.

Results and future work. I proved the Griggs–Yeh Conjecture for 3-regular Hamiltonian graphs [33]. The proof is rather intricate, and requires the study of structural properties of the involved graphs. It starts by pre-labeling G to produce a graph H of ‘badly’ labeled pairs of vertices, and then it uses the structure of H to reduce finding $L(2,1)$ -labeling of G to finding an ordinary coloring of H satisfying some additional constraints.

I also studied $L(2,1)$ -labeling of two special graphs, the incidence graph of the projective plane of order q , and the Kneser graph. These graphs are very interesting and frequently occur in a variety of problems. Since they have a rich structure and it is not trivial to analyze most graph parameters for them, doing so can lead to insights into the more general problem.

The Kneser graph $K(m, k)$ is the disjointness graph on the k -subsets of $\{1, 2, \dots, m\}$. For $K(2k+1, k)$, in [34] I showed that $\lambda(G) \leq 4k + 2$. Here, the $L(2,1)$ -labeling is obtained from a classification of structures between and within the color classes of a special vertex coloring.

For the incidence graph G of the projective plane $PG(2, q)$, Füredi and I [23] show that $\lambda(G) = q^2 + q = \Delta^2 - \Delta$. (The problem was also studied in [40]) To prove this result, we considered packing bipartite graphs into a complete bipartite graph and proved a sufficient condition for such a packing, which is analogous to the result of Sauer and Spencer [52] for packing graphs into a complete graph. For given bipartite graphs G_1 and G_2 with bipartitions X_1, Y_1 and X_2, Y_2 , respectively, a *packing of G_1 and G_2 into $K_{m,n}$* maps $X_1 \rightarrow [m], Y_1 \rightarrow [n]$ and $X_2 \rightarrow [m], Y_2 \rightarrow [n]$ injectively such that $E(G_1) \cap E(G_2) = \emptyset$. We showed that $2\Delta(G_1)\Delta(G_2) < 1 + \max\{m, n\}$ is a sufficient condition for

such a packing.

The result on 3-regular Hamiltonian graphs is the significant progress towards the Griggs and Yeh conjecture. However, the extra condition of Hamiltonicity needs to be removed (to complete the proof for 3-regular graphs). A Hamiltonian cycle can be thought as a 2-factor consisting of one cycle. I am currently working on this with D. West by considering 3-regular graphs with 2-factors consisting of arbitrarily many cycles. Note that every 2-edge-connected 3 regular graph has a 2-factor.

We are also working on extending the ideas from the incidence graph of $PG(2, q)$ to a more general class of bipartite graphs. Füredi and I have succeeded in classifying the case $\Delta = 3$ and are pursuing other cases.

7 Rectilinear equilateral sets in \mathbb{R}^n

Background. This is a 20-year-old problem in combinatorial geometry. Kusner [28] conjectured that the maximum size of a set whose elements are pairwise equidistant under the ℓ_1 -norm in \mathbb{R}^n is $2n$. If it is true, this would be sharp. The conjecture has been proved for $n = 3$ [7] and $n = 4$ [41]. Recently, Alon and Pudlak [5] gave an upper bound $O(n \ln n)$. These proofs are very involved and use ideas from embeddings of metric spaces, approximation theory, etc. However, in the words of the authors, their methods won't solve the full conjecture.

Results. Since this problem has the same flavor as that of computing $\chi(\mathbb{R}_1^n, 1)$, I have worked on it as well in [35].

I have converted the problem into a certain coloring of the sphere. It was shown that the Kusner conjecture is true if and only if any equidistant set $X \subset \mathbb{R}^d$ with $|X| > \frac{3d}{2}$ contains more than $\frac{3d}{2}$ points on a unit ℓ_1 -sphere in \mathbb{R}^d . Related to this, I have shown that the maximum size of an equidistant set on the surface of the unit generalized-octahedron (i) without its extreme point is at most n , and (ii) without two opposite vertices is at most $\frac{3n}{2}$, and both are sharp.

I have also related this problem to the piercing (transversal) number of convex bodies in \mathbb{R}^n . For a given family of (at least three) pairwise intersecting ℓ_1 -spheres, what is the maximum number s such that any s of them in the family touch simultaneously. With an elementary method, I have shown that $s \geq 3$ and the intersection of three spheres is a single point. It had also been shown in [41], in which arguments in embeddings of metric spaces and partially ordered sets are used.

Finally, we have shown that in \mathbb{R}^n there exists such a family unit generalized octahedron in which the centers of the unit ℓ_1 -spheres lie in a convex polytope with at most $2n$ facets. More precisely,

Theorem 6 *Let X be an equidistant set in \mathbb{R}^n and let $\mathbf{a}, \mathbf{b} \in X$. For each $\mathbf{u} \in X \setminus \{\mathbf{a}, \mathbf{b}\}$, if $\mathbf{c}^{\mathbf{x}} \in \mathbb{R}^n$ is the center of the ℓ_1 -unit sphere that contains the three points \mathbf{a}, \mathbf{b} and \mathbf{u} , then the affine hull of the set $\{\mathbf{c}^{\mathbf{x}} \mid \mathbf{u} \in X \setminus \{\mathbf{a}, \mathbf{b}\}\}$ is the hyperplane orthogonal to the sign vector of $\mathbf{b} - \mathbf{a}$ and passing through the point $\frac{\mathbf{a} + \mathbf{b}}{2}$.*

This theorem yields another proof of the Kusner's conjecture for $n = 3$, which was also proved in [7].

Future work. The $s = 3$ above is the best for $n \geq 3$ by the set $X_0 = \{\mathbf{e}_i : 1 \leq i \leq n\} \cup \{\mathbf{p} := (\frac{1}{n-2}, \dots, \frac{1}{n-2}) \in \mathbb{R}^n\}$; The four points $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{p}\}$ do not belong to one ℓ_1 -sphere. But, the size of X_0 is small ($|X_0| = d + 1$). So, if the size of an equidistant set is large (such as $2n$ or even $n + 2$), we hope that the value s can be improved – hopefully $s = 1$, which would be enough to prove the Kusner conjecture.

Theorem 6 was used to show the case $n = 3$ in [35]. I am hoping that this theorem and other results mentioned together can be used for further cases and I am working on it now.

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