

Summary of Papers

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Publications

- Published

1. **Distance graphs from p -adic norms.** *Integers–Electronic Journal of Combinatorial Number Theory*, 10 (2010) 379–392. (with H. Maharaj)

A distance graph is a graph (S, D) with the set of points S in a metric space as vertex set and with an edge joining two vertices u and v if and only if $\|u - v\| \in D$ where D is a prescribed subset of the positive reals. When $S = \mathbb{R}^n$ (n -dimensional reals) and $D = \{1\}$ (unit-distance) under ℓ_p -norm, it is the well-known Hadwiger–Nelson problem that I have worked on.

In this paper, we consider integer distance graphs (\mathbb{Z}, D) under the Euclidean, which were first systematically studied by Eggleton–Erdős–Skilton in 1985. One of main goals in these problems is characterizing prescribed distance sets D having finite chromatic number. Ruzsa, Tuza, and Voigt gave a sufficient condition for $\chi(G(\mathbb{Z}, D))$ to be finite in 2002. Maharaj and I approach the problem from the viewpoint of p -adic norms. We use p -adic norm to derive results on Euclidean distance graphs. We have given bounds on the chromatic number that depend on the divisibility properties of the distances d_i in D . Also, we have given a conditional characterization that complements Ruzsa, Tuza, and Voigt’s result. Our strongest result is as follows. Let p_1, p_2, \dots, p_n be distinct prime numbers, and let $\Lambda \subset \mathbb{N}_0^n$. If $D := \{ap_1^{e_1}p_2^{e_2} \dots p_n^{e_n} : (e_1, e_2, \dots, e_n) \in \Lambda, a \in \mathbb{Z} \text{ with } p_i \nmid a \text{ for all } 1 \leq i \leq n\}$, then the distance graph $G(\mathbb{Z}, D)$ has infinite chromatic number iff the exponent set Λ contains a strictly increasing sequence, where an order $(e_1, e_2, \dots, e_n) < (e'_1, e'_2, \dots, e'_n)$ in Λ is defined if $e_i < e'_i$ for each $1 \leq i \leq n$.

2. **Covering n -space by convex bodies and its chromatic number.** *Discrete Mathematics, Special issue in honor of M. Simonovits*, 308 (2008), 4495–4500. (with Z. Füredi)

* Referenced in the Springer monograph ‘Research Problems in Discrete Geometry’ by Brass, Moser, Pach.

Rogers (1957) proved that for every closed convex body C in \mathbb{R}^n , there is a covering of \mathbb{R}^n by translates of C that has density at most $O(n \ln n)$. However, a covering with low global density can have high multiplicity, where the multiplicity is the maximum number of copies of C covering a single point. Erdős and Rogers (1962) showed that, for sufficiently large n , there is a covering of \mathbb{R}^n by translates of C that has density at most $O(n \ln n)$ and multiplicity at most $O(n \ln n)$. In this paper, we give a new combinatorial proof of this using the Lovasz Local Lemma.

We use the above covering to obtain $\chi(\mathbb{R}_p^n, 1) \leq c(n \ln n)5^n$, independent of p .

3. **$L(2,1)$ -labeling for Hamiltonian graphs of maximum degree 3.** *SIAM Journal on Discrete Mathematics*, 22 (2008), 213–230.

* Led to an invited talk at the Workshop on Extremal Combinatorics at DIMATIA Institute, Charles University, Czech Republic.

A nonnegative-integer coloring f of the vertices of a graph G is an $L(2,1)$ -labeling if $|f(u) - f(v)| \geq 2$ for each edge uv and $|f(u) - f(v)| \geq 1$ for each pair $u, v \in V(G)$ at distance 2. The $L(2,1)$ -labeling span of G , denoted by $\lambda(G)$, is the smallest number t such that G has an $L(2,1)$ -labeling using no label larger than t . Griggs and Yeh (1992) conjectured that always $\lambda(G) \leq (\Delta(G))^2$, which has been shown only for some specific graph families. We prove this for graphs of maximum degree 3 with a spanning cycle.

4. **Distance graph on \mathbb{Z}^n with ℓ_1 -norm.** *Theoretical Computer Science, Special issue on Combinatorics of the Discrete Plane and Tilings*, 319 (2004), 357–366. (with Z. Füredi)

* Referenced in the Springer monograph ‘Research Problems in Discrete Geometry’ by Brass, Moser, Pach.

A long-standing open problem in combinatorial geometry is the chromatic number of the unit-distance graph in \mathbb{R}^n ; here points are adjacent if their distance in the ℓ_2 norm is 1. For $n = 2$, we know the answer is between 4 and 7. In this paper, we consider the analogous problem on the discrete space \mathbb{Z}^n , n -dimensional integer lattice, with two points adjacent if they are at distance r , under the ℓ_1 -norm. Let the chromatic number of this graph be $\chi(\mathbb{Z}^n, r)$. The graph (\mathbb{Z}_1^n, r) is bipartite when r is odd. We study $\chi(\mathbb{Z}_1^n, r)$ for even r . We prove that $\chi(\mathbb{Z}^n, 2) = 2n$ for all n , but $\chi(\mathbb{Z}^n, r) \geq 2n + 1$ for $n \geq 3$ and even $r \geq 4$. In addition, we give exponential bounds $(1.067)^n \leq \chi(\mathbb{Z}^n, r) \leq \frac{1}{\sqrt{2\pi n}}(5e)^n$ for all n and even r . For small values of r , we prove a more useful upper bound, $3r^{n-2}$ for all n and all even $r \geq 4$.

We also consider the graph on \mathbb{R}^n in which points are adjacent if their distance is r in the ℓ_p -norm for $1 \leq p \leq \infty$, and denote its chromatic number by $\chi(\mathbb{R}_p^n, r)$. By scaling, $\chi(\mathbb{R}_p^n, r) = \chi(\mathbb{R}_p^n, 1)$. We prove that $(1.067)^n \leq \chi(\mathbb{R}_p^n, 1) \leq \sqrt{p/(2\pi n)}(5(ep)^{1/p})^n$ for all n .

- Submitted

1. **Rectilinear equilateral sets in \mathbb{R}^n** , manuscript.

Kusner (1983) conjectured that the maximum size of a set whose elements are pairwise equidistant under the ℓ_1 -norm in \mathbb{R}^n is $2n$. If true, this would be sharp. The conjecture has been proved for $n = 3$ (in 1998) and $n = 4$ (in 2000). Recently, Alon and Pudlak gave an upper bound of $O(n \ln n)$. These proofs are very involved and use ideas from embeddings of metric spaces, approximation theory, etc. However, in the words of the authors, their methods won’t solve the full conjecture.

We show that the Kusner conjecture is true if there exists an equidistant set of size $3n/2$ on the surface of the generalized-octahedron with radius 1 in \mathbb{R}^n . Related to this, we show that the maximum size of an equidistant set on the surface of the unit generalized-octahedron without its extreme point is at most n , and this is sharp. Also, we show that if \mathcal{F} is a family of the unit generalized-octahedrons such that every two members of \mathcal{F} intersect, then every three members of \mathcal{F} must intersect. We are working on extending this result to showing that a same \mathcal{F} with at least $2n + 1$ members must be an intersecting family. This would be enough to prove the Kusner conjecture.

Work in progress

- Soon to be submitted

1. **Distance graph on \mathbb{Z}^n with several distances under ℓ_1 -norm**, (*tentative title*), (with H. Maharaaj and S. Anderson)

- Preprints (Will be submitted after improvement)

1. **On largeness and accessibility**, manuscript.

Let D be a prescribed subset of the integer set \mathbb{Z} . If any r -coloring of $\mathbb{Z}^+ \cup \{0\}$ yields an arbitrarily long monochromatic arithmetic progression whose common difference d belongs to D , we say that D is *r-large*. If any r -coloring of $\mathbb{Z}^+ \cup \{0\}$ yields an arbitrarily long monochromatic sequence of distinct integers whose gaps belong to D , the set D is called *r-accessible*. Landman and Robertson ask to determine the largest number r such that

the prime set P is r -accessible, and the same question for its translate $P + c$. It turns out difficult to determine whether P is 2-accessible or not. In this paper, we investigate the largeness and the accessibility of D_m and $D_m + c$ where D_m is all the integers that is coprime to a fixed integer $m \geq 2$ and $c \geq 1$. We give bounds on those, and in particular, we show that $D_m + c$ is r -large, and consequently r -accessible, for all $r \geq 1$ when $\gcd(m, c) = 1$.

2. **Distance graphs on the integer line.** (with H. Maharaj)

For a complete characterization of Euclidean distance graphs on the integer line described in the item 1 above, we have pursued further on the problem using distribution of fractional part of real numbers. We obtain a bit stronger result than in the item 1.

3. **On the structure of p -adic distance graphs.** (with H. Maharaj)

4. **L(2,1)-labeling and packing of bipartite graphs.** (with Z. Füredi)

For the incidence graph G of the projective plane $PG(2, q)$, we show that $\lambda(G) = q^2 + q = \Delta^2 - \Delta$. To prove this result, we consider packing bipartite graphs into a complete bipartite graph and prove a sufficient condition for such a packing, which is analogous to the result of Sauer and Spencer (1978) for packing graphs into a complete graph.

5. **L(2,1)-labeling for Kneser Graphs.**

The Kneser graph $K(m, k)$ is the disjointness graph on the k -subsets of $\{1, 2, \dots, m\}$. For $G := K(2k + 1, k)$, we show that $\lambda(G) \leq 4k + 2$, which also implies the best known bound, $4k + 2$, on $\chi(G^2)$, where G^2 is the graph on $V(G)$ with two vertices adjacent if they are at distance one or two in G .

6. **Security number of Kneser graphs.**

A subset $S \subset V$ is a *secure* set if $|N[U] \cap S| \geq |N[U] \setminus S|$ for every subset $U \subset S$, where $N[U] = U \cup N(U)$. We seek the parameter $\rho(G)$, the *security number* of G , that is the minimum size of a secure set in G . This is introduced by Brigham, Dutton, and Hedetniemi (2005) as a generalization of defensive alliances in graphs. Observe that $\rho(K_n) = \lceil n/2 \rceil$. They asked whether $\lceil n/2 \rceil$ is an upper bound for $\rho(G)$ for every graph G , and, in particular, for Kneser graphs $K(m, k)$ where $n = \binom{m}{2}$.

We prove that $\rho(K(m, 2)) = \lceil \frac{n+1}{2} \rceil$, a negative answer to their question when $k = 2$. We are currently working on extending this result to $K(m, k)$.

- In preparation (have been discussed, currently paused, but to be continued)

1. **Distance graphs on Gaussian integers** (with H. Maharaj)

Another natural generalization of the distance graph problem on \mathbb{Z} is as follows. We use a specific example to illustrate the idea. Let $K = \mathbb{Q}(i)$ where $i^2 = -1$ and let $R = \mathbb{Z}[i]$. Then R is the ring of integers of K , that is, R is the integral closure of \mathbb{Z} in K . As an Abelian group, R is isomorphic to \mathbb{Z}^2 and geometrically R is a lattice in the plane. A prime p of \mathbb{Z} gives rise to two p -adic norms on R if $p \equiv 1 \pmod{4}$, otherwise p gives rise to a single p -adic norm on R . The product formula $|x| \prod_p \|x\|_p = 1$ still holds, so specifying an Euclidean norm is tantamount to specifying several p -adic norms. The same type of questions considered in this section can be asked about distance graphs on R .

In general, if R is a ring of integers in a number field K , then it is possible for R to have several inequivalent archimedean norms. An example of this would be the ring of integers of the field $\mathbb{Q}(\sqrt[3]{2})$ which has two complex embeddings and one real embedding so there are two inequivalent Euclidean norms. It would interesting to study how the structure of the distance graphs would change when one transitions from \mathbb{Z} to general rings of integers. It is hoped that by studying such generalisations, light would be shed on the standard distance graph problem on \mathbb{Z} . Furthermore, it is known that if K is a degree n extension of \mathbb{Q} then the ring R is a free \mathbb{Z} - module of rank n . So studying distance graphs on R is tantamount

to studying a class of distance graphs on the lattice \mathbb{Z}^n . In the next section we consider a class of distance graphs on \mathbb{Z}^n with respect to the ℓ_1 norm.

2. **L(2,1)-labeling for 3-regular graphs** (with D. West)

The result on Hamiltonian graphs of maximum degree 3 in [3] above is the first significant progress towards the Griggs and Yeh conjecture in the last few years. However, the extra condition of Hamiltonicity needs to be removed to complete the proof for graphs of maximum degree 3. Every 2-edge-connected 3 regular graph has a 2-factor (spanning subgraph consisting of disjoint cycles). A Hamiltonian cycle can be thought as a 2-factor consisting of one cycle. We are making progress towards $L(2,1)$ -labeling of 3-regular graphs with an arbitrary 2-factor.

3. **The persistence of a number**

In the sequence 679, 368, 168, 48, 32, 6, each term is the product of the decimal digits of the previous one. Neil Sloane (1973) defines the *persistence* of a number as the number of steps before the number collapses to a single digit. In the ternary representation, the second term is zero or a power of 2. It is conjectured that all powers of 2 greater than 2^{15} contain a zero when written in ternary. The importance of this conjecture is that the maximum persistence in the ternary representation is 3.

We have proved that at least half of the powers of 2 contain zero in their ternary representations. It is proved by considering only special ternary representations – consecutive 1s and 2s. We are working on extending the method to arbitrary appearances of 1s and 2s to complete the conjecture.