

Coloring metric spaces, and $L(2, 1)$ -labelling of graphs

Abstract of Ph.D Thesis

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This thesis focuses on topics in combinatorial geometry and extremal graph theory. It contains results on coloring geometric structures and graph coloring extensions.

We consider coloring n -space under different metrics. In 1955, Hadwiger asked: What is the minimum number of colors needed for coloring the Euclidean plane so that no two points at unit distance receive the same color? The answer is known to be between 4 and 7, with no improvement in the last 50 years. The same question can be asked for any metric space. This has turned out to be a challenging problem with very little progress made on it. We consider ℓ_p -spaces and investigate $\chi(\mathbf{R}_p^n, 1)$, i.e., the chromatic number of the unit distance graph on \mathbf{R}^n under the ℓ_p -norm. Almost nothing is known about these chromatic numbers except for some results on the Euclidean space. By studying certain fundamental geometric ideas, we proved the upper bounds, $\sqrt{p/(2\pi n)}(5(ep)^{1/p})^n$ (for all n), 9^n (for all n), and $c(n \ln n)5^n$ (for large n) on $\chi(\mathbf{R}_p^n, 1)$ for general $p \geq 1$. We construct a finite subgraph of $(\mathbf{R}_p^n, 1)$ with small independence number and, consequently, large chromatic number, which gives a lower bound of $(1.139)^n$ for all p .

Another natural candidate for metric space coloring is the discrete space \mathbf{Z}^n . In this case, the ℓ_1 -norm is the reasonable metric and, since distances cannot be scaled to one, we consider the graph (\mathbf{Z}_1^n, r) on \mathbf{Z}^n with two points adjacent if their ℓ_1 -distance is a fixed integer r . Since the graph (\mathbf{Z}_1^n, r) is bipartite when r is odd, we study $\chi(\mathbf{Z}_1^n, r)$ for even r . In this thesis, we prove that $\chi(\mathbf{Z}_1^n, 2) = 2n$ for all n , but $\chi(\mathbf{Z}^n, r) \geq 2n + 1$ for $n \geq 3$ and even $r \geq 4$. In addition, we give an exponential lower bound $(1.139)^n$ by constructing an appropriate finite subgraph. Since (\mathbf{Z}_1^n, r) is a subgraph of (\mathbf{R}_1^n, r) , we have $\chi(\mathbf{Z}_1^n, r) \leq \chi(\mathbf{R}_1^n, r) = \chi(\mathbf{R}_1^n, 1)$, so the upper bounds for $\chi(\mathbf{R}_1^n, 1)$ hold here. For small values of r , we have a more useful upper bound, $3r^{n-2}$ for all n and all even $r \geq 4$.

The third upper bound $c(n \ln n)5^n$ on $\chi(\mathbf{R}_p^n, 1)$, mentioned above, uses a special covering of the n -space with convex bodies. Roughly speaking, the *density* of a covering of \mathbf{R}^n is the average number of bodies needed to cover a given unit space. In 1957, Rogers proved that, for a given closed convex body C in the n -dimensional Euclidean space, where $n \geq 3$, there is a covering for \mathbf{R}^n by translates of C with density $c(n \ln n)$ for an absolute constant c . However, low density does not imply low multiplicity of the covering, where *multiplicity* is the number of copies of $C \in \mathcal{C}$ containing each point. Even though the global density of a covering is low, there can exist local clusters of high multiplicity. Later in 1962, Erdős and Rogers showed that, for sufficiently large n , there exists such a covering with not only density at most $cn \ln n$ but also multiplicity at most $c'n \ln n$ for an absolute constant c' . In this thesis, we give a new combinatorial proof of this covering result using methods from probabilistic combinatorics and discrete geometry.

Finally, we study the $L(2, 1)$ -labelling problem for graphs. F.S. Roberts proposed a variation of the channel assignment problem, which Griggs and Yeh introduced in 1992 and called the

L(2,1)-labelling problem. Namely, a nonnegative-integer coloring f of the vertices of a graph G is an *L(2,1)-labelling* if $|f(u) - f(v)| \geq 2$ for each edge uv and $|f(u) - f(v)| \geq 1$ for each pair $u, v \in V(G)$ distance 2 apart. This extends the usual notion of graph coloring. The *L(2,1)-labelling span* of G , denoted by $\lambda(G)$, is the smallest number t such that G has an *L(2,1)-labelling* using no label larger than t .

Griggs and Yeh (1992) conjectured that always $\lambda(G) \leq (\Delta(G))^2$. This has been a motivating problem for research in this field. Note that it is enough to consider connected, Δ -regular graphs. Even the case $\Delta = 3$ has remained open. We prove this conjecture for 3-regular Hamiltonian graphs.

We also show that $\lambda(G) = q^2 + q = \Delta^2 - \Delta$ for the incidence graph G of the projective plane $PG(2, q)$. To prove this result, we considered packing bipartite graphs into a complete bipartite graph and proved a sufficient condition for such a packing, which is analogous to the result of Sauer and Spencer (1978) for packing graphs into a complete graph. For given bipartite graphs G_1 and G_2 with bipartitions X_1, Y_1 and X_2, Y_2 , respectively, a *packing of G_1 and G_2 into $K_{m,n}$* maps $X_1 \rightarrow [m], Y_1 \rightarrow [n]$ and $X_2 \rightarrow [m], Y_2 \rightarrow [n]$ injectively such that $E(G_1) \cap E(G_2) = \emptyset$. We showed that $2\Delta(G_1)\Delta(G_2) < 1 + \max\{m, n\}$ is a sufficient condition for such a packing.

Other than that, we study the *L(2,1)-labelling* of the Kneser graph $K(n, k)$, the disjointness graph on the k -subsets of $\{1, 2, \dots, n\}$. For $G = K(2k + 1, k)$, we show that $\lambda(G) \leq 4k + 2$.