Solution to Practice Exam 2, Math 3303, Spring 2007

Print Your Name: ________________________________

Direction: Show all your work. Work without sufficient detail will not get full credits.

1. a) Find the general solution of $y'' + 2y' + y = 0$.

Solution: Auxiliary Equation: $m^2 + 2m + 1 = 0$, so $(m+1)^2 = 0$. Thus, $m_1 = -1 = m_2$. Hence, the general solution is

$$y = C_1 e^{-x} + C_2 x e^{-x}.$$

b) Using the method of undetermined coefficients, find a particular solution of $y''+2y'+y = \sin x$.

Solution: The associated homogeneous equation is the one in part a) above.
So from a) we know $\sin x$ and $\cos x$ are not solutions to the homogeneous equation. Hence, we set

$$y_p = A\cos x + B\sin x.$$

Then $y_p' = -A\sin x + B\cos x$ and $y_p = -A\cos x - B\sin x$.

$$\begin{aligned}\sin x &= y_p'' + 2y_p' + y_p \\ &= (-A\cos x - B\sin x) + 2(-A\sin x + B\cos x) + (A\cos x + B\sin x) \\ &= -2A\sin x + 2B\cos x.\end{aligned}$$

Thus, $1 = -2A$ and $0 = 2B$, and hence $A = -\frac{1}{2}$ and $B = 0$. Therefore, $y_p = -\frac{1}{2}\cos x$.

2. (3 points) Determine the number of solutions that the boundary value problem $y'' + y = 0$, $y(0) = 0$, $y'\left(\frac{\pi}{2}\right) = 0$ has. (7 points) Justify your answer.

Solution: The auxiliary equation of $y''+y = 0$ is $m^2+1 = 0$. So $m^2 = -1$ and $m = \pm\sqrt{-1} = \pm i$ ($\alpha = 0$, $\beta = 1$).
Thus, the general solution of $y'' + y = 0$ is $y(x) = C_1\cos x + C_2\sin x$.

$$0 = y(0) = C_1\cos 0 + C_2\sin 0 = C_1.$$

Since $C_1 = 0$, $y(x) = C_2\sin x$ and $y'(x) = C_2\cos x$.

$$0 = y'\left(\frac{\pi}{2}\right) = C_2\cos\left(\frac{\pi}{2}\right) = C_2\cdot 0 = 0 \quad \text{whatever } C_2 \text{ is.}$$

Thus, $y(x) = C_2\sin x$ is the solution to the boundary value problem for each and every real number C_2. Therefore, there are infinitely many solutions.

3. Find the general solution of $y'' - y = e^{2x}$.

Solution: The auxiliary equation to the associated homogeneous equation $y'' - y = 0$ is $m^2 - 1 = 0$, and hence, $m^2 = 1$. So $m = \pm 1$, two distinct real numbers. Thus, the complementary solution is

$$y_c = C_1e^{-x} + C_2e^x.$$

Next, the correct form of a particular solution is $y_p = Ae^{2x}$. So $y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$. Thus,

$$\begin{aligned} e^{2x} &= y_p'' - y_p \\ &= 4Ae^{2x} - Ae^{2x} \\ &= 3Ae^{2x}. \end{aligned}$$

So $3A = 1$ and hence, $A = \frac{1}{3}$ and $y_p = \frac{1}{3}e^{2x}$. Therefore, the general solution is

$$y = y_c + y_p = C_1e^{-x} + C_2e^x + \frac{1}{3}e^{2x}.$$

4. Find the general solution of $y'' - y = xe^x$.

Solution: From Problem 3, we have that the complementary solution is

$$y_c = C_1e^{-x} + C_2e^x.$$

Next, our first choice of a particular solution is $y_p = (Ax + B)e^x$. Here, Be^x is a solution to the associated homogeneous equation (i.e., it overlaps with the complementary solution). So $y_p = (Ax^2 + Bx)e^x$ is the correct form of a particular solution.

$$\begin{aligned} y_p &= (Ax^2 + Bx)e^x, \\ y_p' &= (2Ax + B)e^x + (Ax^2 + Bx)e^x = (Ax^2 + (2A + B)x + B)e^x, \\ y_p'' &= (2Ax + (2A + B))e^x + (Ax^2 + (2A + B)x + B)e^x = (Ax^2 + (4A + B)x + 2A + 2B)e^x. \end{aligned}$$

So

$$xe^x = y_p'' - y_p = (Ax^2 + (4A + B)x + 2A + 2B)e^x - (Ax^2 + Bx)e^x = (4Ax + 2A + 2B)e^x.$$

We get $1 = 4A$ and $2A + 2B = 0$. Thus, $A = \frac{1}{4}$ and $B = -\frac{1}{4}$, and $y_p = \left(\frac{1}{4}x^2 - \frac{1}{4}x\right)e^x$. Therefore, the general solution is

$$y = y_c + y_p = C_1e^{-x} + C_2e^x + \left(\frac{1}{2}x^2 - \frac{1}{4}x\right)e^x.$$

5. Find the correct form of a particular solution y_p when you solve $y'' - y = xe^x + \sin x$ by the method of undetermined coefficients. *You do not need to solve this equation.*

Solution: The general solution of the associated homogeneous equation $y'' - y = 0$ is $y_c = C_1e^{-x} + C_2e^x$.

A first choice of y_p is

$$y_p = (Ax + B)e^x + C\cos x + D\sin x.$$

Here e^x in the first term is a solution of the homogeneous equation $y'' - y = 0$. So our second choice is

$$y_p = x \cdot (Ax + B)e^x + C\cos x + D\sin x = (Ax^2 + Bx)e^x + C\cos x + D\sin x.$$

Now none of the terms above is a solution of the homogeneous equation $y'' - y = 0$ and hence, the correct form of y_p is

$$y_p = (Ax^2 + Bx)e^x + C\cos x + D\sin x.$$

6. Solve $y'' + y = \sec^2 x$.

Solution: Here we have to use the method of variation of parameters. Since the coefficient of y'' is 1, $f(x) = \sec^2 x$.

Next, we solve the associated homogeneous equation to find y_1 and y_2. The auxiliary equation of the associated homogeneous equation $y'' + y = 0$ is $m^2 + 1 = 0$. So $m^2 = -1$ and $m = \pm\sqrt{-1} = \pm i$ ($\alpha = 0$, $\beta = 1$).

Thus, the general solution of $y'' + y = 0$ is $y(x) = C_1 \cos x + C_2 \sin x$. We take $y_1 = \cos x$ and $y_2 = \sin x$.

$y_1 = \cos x$, $y_2 = \sin x$ are two linearly independent solutions to the associated homogeneous equation. So there is a particular solution of the form $y_p = u_1 y_1 + u_2 y_2$, where

$$u_1' = \frac{-y_2 f}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-\sin x \sec^2 x}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \sec^2 x}{1} = \frac{-\sin x}{\cos^2 x},$$

where we used $\sin^2 x + \cos^2 x = 1$ and $\sec x = \frac{1}{\cos x}$. Thus, $u_1 = \int \frac{-\sin x}{\cos^2 x} dx = -\frac{1}{\cos x} + C_3$ where we chose the integral constant C_3 to be 0.

Also,

$$u_2' = \frac{y_1 f}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\cos x \sec^2 x}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\sec x}{1} = \sec x.$$

Thus, $u_2 = \int \sec x dx = \ln|\sec x + \tan x| + C_4$ where we chose the integral constant C_4 to be 0.

Therefore,

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{\cos x}(\cos x) + (\ln|\sec x + \tan x|)(\sin x) = -1 + (\ln|\sec x + \tan x|)(\sin x).$$

Finally, the general solution is

$$y = y_c + y_p = C_1 \cos x + C_2 \sin x - 1 + (\ln|\sec x + \tan x|)(\sin x).$$

7. Show that 1 and $\tan x$ are linearly independent.

Solution:

$$\begin{vmatrix} 1 & \tan x \\ 0 & \sec^2 x \end{vmatrix} = \sec^2 x \neq 0, \quad \text{for example, for } x = 0.$$

Hence, 1 and $\tan x$ are linearly independent.

8. One knows the differential equation $y'' - 2y' + y = 0$ has two linearly independent solutions $y_1 = e^x$ and $y_2 = xe^x$. Use the reduction of order to derive $y_2 = xe^x$ from $y_1 = e^x$.

You should begin by setting $y_2 = uy_1 = ue^x$. Then using the reduction of order, show that you could choose $u = x$. *If you do not use the reduction of order, you will get no credit.*

Solution: Set $y_2 = y_1 u = ue^x$. Then $y_2' = u'e^x + ue^x$ and $y_2'' = u''e^x + 2u'e^x + ue^x$. So

$$\begin{aligned} 0 &= y_2'' - 2y_2' + y_2 \\ &= u''e^x + 2u'e^x + ue^x - 2(u'e^x + ue^x) + ue^x \\ &= u''e^x. \end{aligned} \tag{1}$$

And $u'' = 0$. Now set $u' = a$ and $u = ax + b$. So by taking $a = 1$ and $b = 0$, we get $u = x$ and hence, $y_2 = ue^x = xe^x$.

Note: Note that this is relatively easy case because we have $u'' = 0$ after equation (1). In general, one should expect a first order (more complicated) linear equation in w with $w = u'$.