RESEARCH STATEMENT

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My research interests lie in differential equations and complex analysis, and their applications to mathematical physics. In particular, I have worked on the spectral theory of one-dimensional Schrödinger operators associated with the differential expression $-\frac{d^2}{dx^2} + V(x)$, with either polynomial potentials V (anharmonic oscillators), periodic potentials V (Hill operators), or algebro-geometric potentials. (An algebro-geometric potential is a solution of one of the stationary KdV equations.) Even though my work includes self-adjoint Hamiltonians as special cases, I am mostly fascinated with properties of non-self-adjoint operators.

Anharmonic Oscillators

The theory of anharmonic oscillators has important applications in all fields of quantum physics as well as in fields of molecular, nuclear, and condensed matter physics. This is why their properties (in particular, those related to their spectrum) have been the focus of attention ever since the birth of quantum mechanics. Anharmonic oscillators serve as simple model systems in these fields.

The spectrum of an anharmonic oscillator consists of pure points (the energy levels or eigenvalues). The anharmonic oscillator has infinitely many eigenvalues λ_n [27], but except for the harmonic oscillator, there is no formula for the eigenvalues *yet*. Thus, for a given anharmonic oscillator, one would like to approximate the eigenvalues, to find asymptotic expansions of these eigenvalues λ_n $(n \to \infty)$, and to study how the eigenvalues depend on the potential. Also, I am interested in the inverse spectral problem, recovering the polynomial potential from the asymptotics of the eigenvalues and eigenvalue counting functions.

Background: In 1933, G. D. Birkhoff [5] formulated the eigenvalue problem and explained in general terms why one should study the asymptotics of fundamental solutions in order to compute the eigenvalues, but no definite results were given. In 1966, Evgrafov and Fedoryuk implemented the program of Birkhoff, writing *formal* asymptotic expansions of the solutions (essentially a WKB approximation) and obtained the leading order asymptotic behavior of the eigenvalues (see, e.g., [9]). However, using their approach, it is difficult to compute the next terms in the asymptotics of the eigenvalues.

Moreover, Evgrafov and Fedoryuk tacitly assumed that the point at infinity is not an essential singularity, and hence, their method cannot detect exponentially small terms in the asymptotics. Demonstrating an absence (or presence) of the exponentially small terms of the type e^{-n^a} for some $a > 0$, terms that give no contribution to the asymptotics of the eigenvalues λ_n, has been a major and challenging problem in the theory of anharmonic oscillators. I do believe that there are such exponentially small terms, and this question is closely related with a certain nature of the WKB approximation (see Problem 2 below).

Techniques in perturbation theory give another method for investigating the eigenvalues. Bender and Wu [4] in 1969, and Simon [28] in 1970 studied the eigenvalues of the anharmonic oscillators with the potential $V(x) = x^2+\beta x^4$. They proved the analyticity of the eigenvalues in the coupling constant β. Also they showed that the coefficients of the Rayleigh-Schrödinger perturbation series grow factorially and hence that the series diverges for all $\beta \in \mathbb{C} \setminus \{0\}$. Thus, the βx^4 perturbation of the harmonic oscillator $-\frac{d^2}{dx^2}+x^2$ in $L^2(\mathbb{R})$ is not regular, since the well-developed theory of regular perturbations due to Kato and Rellich would guarantee the convergence of the Rayleigh-Schrödinger series (see, e.g., [13]).

The pioneering work of Bender and Wu [4], and Simon [28] has stimulated research in this direction, resulting in a huge amount of literature on the subject. Some people have applied various summation techniques to the divergent series for a very limited class of anharmonic oscillators, such as the Borel [10] and Stieltjes-Padé [15] methods. However, in order to get information on the eigenvalues, typically people have approximated the solutions of the Schrödinger equation by the phase space or WKB method, following Birkhoff, or even used the Bohr-Sommerfeld asymptotic formula [14], as well as a large number of refinements of these methods used for some particular cases.

My Research: In 2004, I explicitly computed the first $\lfloor (m+2)/2 \rfloor$ terms in the asymptotics of the eigenvalues, where $m \geq 3$ is the degree of the polynomial potential [22, 24, 25]. Two distinctive features of my result are its applicability to every anharmonic oscillator with a complex potential and explicitness of the coefficients of the asymptotics.

To the best of my knowledge, the only other eigenvalue asymptotics that apply to every anharmonic oscillator were obtained by Evgrafov and Fedoryuk [9], who computed the first term only. Some people [11, 16, 29] computed asymptotics of the eigenvalues with an error term better than mine, but applicable only to a limited class of polynomial potentials such as $x^2 + \beta x^{2m}$. Also, since the coefficients of my asymptotics of λ_n are explicit polynomials of the coefficients in the potential, one can compute the asymptotics with very little effort. Moreover, I obtain inverse spectral results, reconstructing the degree and $\lfloor (m+2)/2 \rfloor$ coefficients of the polynomial potentials from the asymptotics of λ_n (under certain unavoidable restrictions).

I have recently made a breakthrough and found a way of explicitly computing the asymptotics of λ_n to <u>all orders</u>. This new idea will bring an important contribution to the theory of anharmonic oscillators. More detail on what I can do with this new idea will be noted below when we discuss future directions.

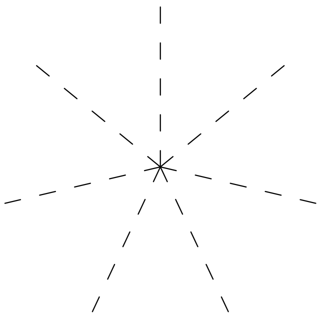

FIGURE 1. Stokes sectors for $m = 5$ and ℓ odd.

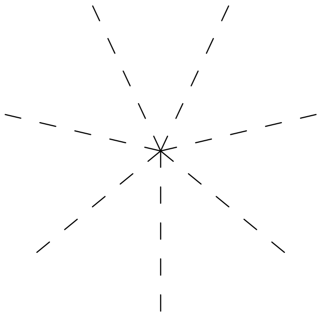

FIGURE 2. Stokes sectors for $m = 5$ and ℓ even.

Anharmonic Oscillators in the Complex Plane: Eigenvalues of a square matrix are the zeros of a polynomial (the characteristic polynomial), whereas the eigenvalues of an anharmonic oscillator are the zeros of a complex analytic function (the spectral determinant or Stokes multiplier of Sibuya [27]). In order to compute the eigenvalues of a square matrix, one finds its characteristic polynomial first. Likewise, to compute the eigenvalues of the anharmonic oscillators, I have studied the Stokes multiplier, in particular, the asymptotic behavior of it near infinity in the complex plane. This leads to my recent results on the asymptotics of the eigenvalues. This idea of using the Stokes multiplier to compute coefficients of the asymptotics of the eigenvalues has been seldomly used.

The anharmonic oscillators on the real line are the most studied and interesting cases, but we will consider a broader class of anharmonic oscillators in the complex plane. The reason for considering these auxiliary anharmonic oscillators is that their Stokes multipliers are connected through some functional equations, which help us investigate the eigenvalues. Also these auxiliary anharmonic oscillators are of independent interest in the light of recent developments in $\mathcal{PT}$-symmetric quantum field theory.

For integers $m \geq 3$ and $1 \leq \ell \leq m-1$, consider the non-self-adjoint eigenvalue problems

$$-u''(z) + \left[(-1)^{\ell}(iz)^m - P(iz)\right] u(z) = \lambda u(z), \text{ for some } \lambda \in \mathbb{C}, \tag{1}$$

where P is a fixed polynomial of degree at most $m-1$, with the boundary condition that

$$u(z) \to 0 \text{ as } z \to \infty \text{ along the two rays defined by } \arg z = -\pi/2 \pm (\ell+1)\pi/(m+2). \tag{2}$$

If a non-constant function u along with a complex number λ solves (1) with the boundary condition (2), then we call u an *eigenfunction* and λ an *eigenvalue (*or *energy level).*

The asymptotic behavior of solutions of (1) near infinity is rather simple. There are the so-called $(m+2)$ Stokes sectors (see Figures 1 and 2). If a solution of (1) decays to zero along a ray going to ∞ in a Stokes sector, then the solution decays to zero exponentially along every ray in the Stokes sector [12, §7.4]. The rays in (2) are centers of Stokes sectors. Thus, (2) represents all possible decaying boundary conditions (after scaling if necessary). Note that the boundary condition (2) for $\ell = \lfloor m/2 \rfloor$ is equivalent to the eigenfunctions decaying along the positive and negative real axes (hence belonging to $L^2(\mathbb{R})$).

Future Directions.

Problem 1. Anharmonic Oscillators: Direct and Inverse Spectral Problems:

My immediate future project is to write down recent progress on the asymptotics of eigenvalues and eigenvalue counting functions of anharmonic oscillators to all orders. As an application I should be able to prove the existence of exponentially small errors, terms giving no contribution to the asymptotics of the eigenvalues.

Also, I expect to obtain a best possible inverse spectral result, the most efficient way of reconstructing the polynomial potential from the asymptotics of the eigenvalues and their counting functions. For the half-line anharmonic oscillators, I should be able to recover the exact polynomial potential from the first few terms in the asymptotics of the eigenvalues and their counting functions, but the least number of terms needed for this purpose is yet to be determined. The full-line case is a bit trickier than the half-line case since, for example, the polynomials $V(x)$, $V(-x)$, and $V(x - x_0)$ are isospectral potentials. So some knowledge of certain coefficients of the polynomial is needed, and it is yet to be determined how much knowledge on the coefficients of the polynomial is needed.

Note that finitely many parameters, such as coefficients and degree of the potential, and the boundary condition, determine infinitely many eigenvalues of anharmonic oscillators. Thus, it will be interesting to see *how* these finitely many parameters and the infinitely many eigenvalues are related. I hope this will lead me to necessary and/or sufficient conditions such that an infinite sequence of numbers can be the eigenvalues of an anharmonic oscillator.

Problem 2. Anharmonic Oscillators: Explicit Exponentially Small Errors:

People have used the WKB method to approximate fundamental solutions and to find their asymptotics. This seems to be good enough for practical uses because an error of, say 1%, is good in many instances. In the theoretical point of view, however, the asymptotic series solution obtained from the WKB approximation do not converge to the actual solution of the differential equation. This is because the WKB approximation tacitly assumed that the point of infinity is not a type of essential singularity, and hence, cannot detect possible exponentially small errors. My approach should construct an algorithm of finding series solutions that converge to the exact solution.

My medium term goal is to find exponentially small error terms explicitly and also to find these error terms to any given precision. This will lead to approximating the eigenvalues λ_n for every fixed n to any given precision.

Problem 3. Anharmonic Oscillators: Nature of a Singular Perturbation:

I plan to study a certain property of a singular perturbation. When we study a perturbation of an operator, typically we consider a perturbation of a known operator like the βx^4 perturbation of the well-known harmonic oscillator. However, the perturbation is singular in this case, and the Rayleigh-Schrödinger series diverges. For the purpose of investigating the eigenvalues, I believe that one should study the αx^2 perturbation of $-\frac{d^2}{dx^2}+x^4$ in $L^2(\mathbb{R})$ whose Rayleigh-Schrödinger series converges for all small $\alpha \in \mathbb{C}$.

I plan to work on computing the radius r_n of convergence of the Rayleigh-Schrödinger series (about $\alpha = 0$), for λ_n as a function of α. I believe that r_n is finite if and only if $\lambda_n(\alpha)$ collides with another eigenvalue at some α_0. Moreover, I believe that r_n is the largest number such that $\lambda_n(\alpha)$ does not collide with any other eigenvalue for $\alpha \in \mathbb{C}$ with $|\alpha| < r_n$. I know that the sequence of radii r_n diverges to infinity. What is not clear is how fast it diverges to infinity.

For $\beta \in \mathbb{C} \setminus (-\infty, 0]$, there is a simple one-to-one correspondence between the eigenvalues of the anharmonic oscillators with the potentials $x^2+\beta x^{2m}$ and $\beta^{-2/(m+1)}x^2+x^{2m}$. Simon [28] showed the analytic continuation of $\lambda_n(\beta)$ of $-\frac{d^2}{dx^2}+x^2+\beta x^4$ has a "global" third-order branch point at $\beta = 0$. I will try to answer whether this is *solely* due to the algebraic branch cut of $\beta^{-2/3}$.

People use techniques in perturbation theory for computing the eigenvalues. My plan is to use information on the eigenvalues for understanding properties of the regular and singular perturbations of the anharmonic oscillators. (Since I do not plan to use perturbation theory for Problems 1 and 2 above, this should be a reasonable approach.) Also, there are a number of other interesting questions about the singularities of the Rayleigh-Schrödinger series.

Problem 4. Hill Operators and Algebro-Geometric Potentials:

I have worked on the local shape of spectra of Hill operators [21], the spectrum of a class of particular operators [19], and the Neumann trace formula [20]. My plan is to determine when a one-dimensional Schrödinger operator becomes a spectral operator of scalar type in the sense of Dunford and Schwartz [8, pp. 1938 and 2242].

When the potential V is periodic and complex-valued, recent work of Gesztesy and Tkachenko presents a necessary and sufficient condition that ensures the Hill operator is a spectral operator of scalar type. I plan to work with Gesztesy on necessary and sufficient conditions under which a one-dimensional Schrödinger operator with a quasi-periodic algebro-geometric potential becomes a spectral operator.

Problem 5. Possible Area of Research with Undergraduates:

Reality of the Eigenvalues of $\mathcal{PT}$-Symmetric Anharmonic Oscillators: Conventional Hamiltonians in quantum mechanics and quantum field theory are self-adjoint. This is because self-adjointness guarantees real eigenvalues and unitarity of the time evolution, and hence conservation of probability. Recently however, the so-called (non-self-adjoint) $\mathcal{PT}$-symmetric Hamiltonians have gathered considerable attention because many such Hamiltonians have real spectra. A $\mathcal{PT}$-symmetric Hamiltonian is a Hamiltonian that is invariant under the product of the $\mathcal{P}$arity reflection ($x \mapsto -x$) and the $\mathcal{T}$ime reversal operation (complex conjugation). Hence, (1) is $\mathcal{PT}$-symmetric if and only if P is real on the real line.

The physicists Bessis and Zinn-Justin around 1992 for $m = 3$, and Bender and Boettcher in 1998 for $m \geq 3$ [2] raised the following remarkable conjecture.

Conjecture. *The eigenvalues of* $-u''(z) - [(iz)^m + \beta z^2]\, u(z) = \lambda u(z)$, $\beta \in \mathbb{R}$, *under the boundary condition* (2) *with* $\ell = 1$, *are all real and positive [even though the potential is complex!].*

A number of physicists, notably Bender, have done ample numerical, asymptotic, and theoretical studies [2, 3, 17] that support the Conjecture. In 2001, Dorey, Dunning and Tateo [7] handled the polynomial potentials of the form $-(iz)^m + \alpha(iz)^{m/2-1}$, $\alpha \in \mathbb{R}$, and proved the $\beta = 0$ version of the Conjecture. In 2002, I extended the polynomial cases of [7] to the general $\mathcal{PT}$-symmetric polynomial potentials with some sign restrictions on their coefficients, and verified the $\beta \neq 0$ version of the Conjecture [18].

However, there are some $\mathcal{PT}$-symmetric anharmonic oscillators that produce a finite number of non-real eigenvalues [1, 6]. So without any further restrictions on the coefficients of the potentials, the following theorem is the most general result one can expect about reality of eigenvalues of $\mathcal{PT}$-symmetric operators with polynomial potentials.

Theorem 1 ([24, Theorem 1.8]). *All* $\mathcal{PT}$*-symmetric anharmonic oscillators have real and positive eigenvalues only, with at most finitely many exceptions.*

I plan to work with undergraduate students familiar with the complex numbers and integration by parts to extend my reality results to more general polynomial potentials and to find a bound on the number and/or magnitudes of non-real eigenvalues so that numerical studies can verify reality of all eigenvalues.

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