ON THE EIGENPROBLEM OF AN IMAGINARY CUBIC OSCILLATOR

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ABSTRACT. We consider the eigenproblem $-u''(z)+iz^3u(z)=\lambda u(z)$ with $u(\pm\infty+0i)=0$. It is known that $|\arg\lambda|\leq\frac{\pi}{5}$, meaning λ lies in the sector of aperture $\frac{\pi}{5}$ centered on the positive real axis. Here we will show that $|\operatorname{Im}\lambda|\leq(7.1)(\operatorname{Re}\lambda)^{2/3}$. So in particular, $\arg\lambda\to 0$ as $|\lambda|\to\infty$. We also obtain explicit estimates even for small $|\lambda|$, such as that if $|\lambda|\geq 19$, then $|\arg\lambda|<\frac{\pi}{6}$.

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1. INTRODUCTION

We are considering the eigenproblem

$$-u''(z)+iz^3u(z)=\lambda u(z)\quad\text{with } u(\pm\infty+0i)=0, \tag{1}$$

which one can interpret as giving the energy levels λ of a cubic oscillator with a purely imaginary coupling constant. This Schrödinger equation has arisen in a number of physical contexts [10, 11, 12, 15] and references therein.

If u along with a complex number λ solves (1), then we call u *an eigenfunction* and λ *an eigenvalue*. Note that if λ is an eigenvalue, then so is the complex conjugate $\overline{\lambda}$ with the corresponding eigenfunction $\overline{u(-\overline{z})}$.

If one replaces the potential iz^3 in (1) by z^2, one gets the harmonic oscillator, which has real eigenvalues. Translating the z-variable in the imaginary direction gives an oscillator with the same real eigenvalues and with potential $(z+ia)^2=z^2-a^2+2iaz$. When $a\in\mathbb{R}-\{0\}$ this potential is non-Hermitian. So some eigenproblems with non-Hermitian complex potentials have real eigenvalues. Also other authors [7, 8, 16] study more complicated non-Hermitian potentials q on certain compact groups and homogeneous spaces such that the entire spectrum of $-\Delta+q$ is the same as that of $-\Delta$, which is certainly real. The problem (1) is of a different nature: it has a very simple potential term, and ample numerical and asymptotic results support that eigenvalues of (1) are all real and positive. However, there is no rigorous proof of this.

In [18], the author shows that $|\arg\lambda|\leq\pi/5$, so that the spectrum is within an angle $\pi/5$ of being positive real. We improve $|\arg\lambda|\leq\pi/5$ in the next theorem.

Theorem 1. *If λ is an eigenvalue of (1) then $|\operatorname{Im}\lambda|<(7.1)(\operatorname{Re}\lambda)^{2/3}$, so that $\arg\lambda\to 0$ as $\lambda\to\infty$ through a sequence of eigenvalues.*

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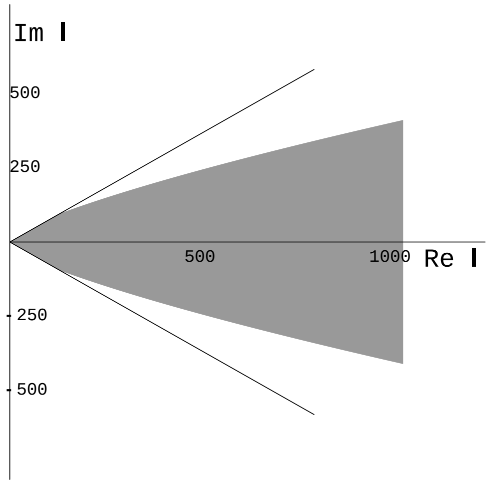

FIGURE 1. The eigenvalues lie in the shaded region, bounded by $|\mathrm{Im}\,\lambda| \lessapprox (7.1)(\mathrm{Re}\,\lambda)^{2/3}$. The rays are $\mathrm{Im}\,\lambda = \pm(\mathrm{Re}\,\lambda)\tan\frac{\pi}{5}$.

More precisely,

$$|\lambda| < R(|\arg\lambda|), \tag{2}$$

where

$$\begin{aligned} R(\theta) \quad := \quad & \left\{ \frac{1}{\log(1+1/\sqrt{2})} \left(\frac{873\cdot 2^{2/3} - 271\sqrt{2}\cdot 3^{1/6}}{330} + \frac{11(\pi+\theta)}{4\cdot 2^{1/3}} \right.\right. \\ + \quad & \left.\left. \frac{1}{880\sin^{5/2}\theta}\left[582\cdot 2^{1/6} + \frac{440(\pi-\theta)-480}{\sin^{11/2}(\pi/6+\theta/3)} + \frac{385(\pi-\theta)-924}{\sin^{5/2}(\pi/6+\theta/3)} \right] \right) \right\}^{6/5}, \\ & for \quad 0 \le \theta \le \pi/5. \end{aligned}$$

The region $\{\lambda : |\lambda| < R(|\arg\lambda|)\}$ *is shaded in Figure 1.*

In proving the above theorem, we bound $|\lambda|$ from above using a very rough estimate of the difference between the eigenfunction and a certain asymptotic expression. Certainly, one can improve this bound when $\arg\lambda$ is "large", say $|\arg\lambda| \ge \pi/6$ or $\pi/10$. We state the contrapositive form of the resulting bounds.

Theorem 2. *Suppose that* λ *is an eigenvalue of (1).*

$$\text{If} \quad |\lambda| \ge 19 \quad \text{then} \quad |\arg\lambda| < \frac{\pi}{6}. \tag{3}$$

$$\text{If} \quad |\lambda| \ge 145 \quad \text{then} \quad |\arg\lambda| < \frac{\pi}{10}. \tag{4}$$

This paper is organized as follows. In the next section, we apply to our problem Hille's asymptotic integration method and the Liouville transformation from the z-plane to a Z-plane. We get that the transformed eigenfunction is asymptotic to $C\sin(Z-\pi/4)$ in a region in the Z-plane that corresponds to a region near the positive imaginary axis in the z-plane. Also we extend the zero-free region of the eigenfunction first obtained in [18].

In Section 3, we investigate the inverse image of the horizontal lines in the Z-plane in order to explicitly estimate, in Section 4, the difference between the eigenfunction and the asymptotic expression. Then in Sections 5 and 6, by estimating the location of the zeros of the eigenfunction, we show that for each $0 < |\arg\lambda| \le \pi/5$, the size of $|\lambda|$ cannot be too big, or else there would be a zero of the eigenfunction in the zero-free region. This gives our main theorems.

Note that the zero boundary conditions in the problem (1) are indeed meaningful because of the following. For each $\lambda \in \mathbb{C}$ there are two linearly independent solutions of (1) without the boundary conditions. Each linear combination of these solutions is either decaying to zero or blowing up exponentially, in any radial direction except for along 5 critical rays, $\arg z = \pi/10, 5\pi/10, 9\pi/10, 13\pi/10, 17\pi/10$, near which the behavior of the solution changes (see Proposition 7). In generic cases, the solution blows up at both $+\infty + 0i$ and $-\infty + 0i$, while in exceptional cases, it can decay to zero as z approaches $+\infty + 0i$ or $-\infty + 0i$. Only in very exceptional cases (when λ is an eigenvalue!) does one find a solution that decays to zero at both $+\infty + 0i$ and $-\infty + 0i$.

2. THE LIOUVILLE TRANSFORMATION AND THE ASYMPTOTIC INTEGRATION

In this section, we will apply Hille's general method to our problem, and get an explicit asymptotic expression for u, and extend the zero-free region of the eigenfunction.

Hille's general method consists of the Liouville transformation and the asymptotic integration. We begin with the Liouville transformation by defining the following. Let

$$Z = \int_{z^*}^{z} \sqrt{\lambda - i\xi^3}\, d\xi + c, \qquad z \in \mathbb{C}. \tag{5}$$

Certainly the map $z \mapsto Z$ is single-valued and analytic on any simply connected domain that does not contain any zeros of $\lambda - iz^3$. We will choose z^* and the constant c in the above definition in Lemma 5. One might consider the integral, in (5) as an indefinite integral and these two constants z^* and c as giving the constant of integration. The purpose of introducing Z is to transform the eigenfunction u to the Z-plane, later in this section. But first, we need to show the map $z \mapsto Z$ is one-to-one on the region $\mathcal{D} = \{z \in \mathbb{C} : \pi/6 < \arg z < \pi/2 + \phi/3\}$. For this we can suppose $0 < \arg\lambda \le \pi/5$, as follows. We know that if λ is an eigenvalue then so is $\overline{\lambda}$. So any result on the location of the eigenvalues for $\arg\lambda > 0$ gives a similar result for $\arg\lambda < 0$. Also in [18], the author shows that $|\arg\lambda| \le \pi/5$. Thus we can assume

$$0 < \phi := \arg\lambda \le \pi/5.$$

Lemma 3. *The map $z \mapsto Z$ is one-to-one in $\mathcal{D}$, provided z^* and the path of the integration are in $\mathcal{D}$.*

Proof. Suppose that there exist z_1 and z_2 in $\mathcal{D}$ with $Z(z_1) = Z(z_2)$. Since $\mathcal{D}$ is convex, it contains the line segment connecting z_1 and z_2. So with the line segment parameterized by $z(t)$, $t_1 \le t \le t_2$, we have

$$0 = Z(z_2) - Z(z_1) = \int_{t_1}^{t_2} \sqrt{\lambda - iz^3(t)}\, z'(t)\, dt.$$

Since $z(t)$ is a line segment, $z'(t) \equiv$ *constant*. Thus we have that

$$\int_{t_1}^{t_2} \sqrt{\lambda - iz^3(t)} dt = 0. \tag{6}$$

But $z(t) \in \mathcal{D}$ and so $\pi < \arg(iz(t)^3) < 2\pi + \phi$. Further, since λ has argument $2\pi + \phi$, we see that $0 < \arg[\lambda - iz(t)^3] < \pi + \phi$. Since

$$\pi < \arg \sqrt{\lambda - iz^3(t)} = \pi + 1/2 \arg[\lambda - iz^3(t)] < 3\pi/2 + \phi/2,$$

it is impossible to get (6). Therefore, the map $z \mapsto Z$ is one-to-one in $\mathcal{D}$. $\square$

Remark. In the proof above, we used $\arg \sqrt{\lambda - iz^3(t)} = \pi + 1/2 \arg[\lambda - iz^3(t)]$. The reason for this choice of the branch of the square root is to have that $z \mapsto Z$ maps a neighborhood of the positive imaginary axis in the z-plane to a neighborhood of the positive real axis in the Z-plane, as we show in the next section.

Lemma 3 shows that the analytic map $z \mapsto Z$ is invertible on its image. We now investigate the map $z \mapsto Z$ further.

Lemma 4. *Let $z(t) \in \mathcal{D}$ be a parameterization of the inverse image of a horizontal line segment in the Z-plane, with* $\operatorname{Re} Z(z(t))$ *increasing (i.e., travese the segment left to right). Then*

$$\arg z'(t) = \pi - 1/2 \arg[\lambda - iz(t)^3] \in (\pi/2 - \phi/2, \pi). \tag{8}$$

From this lemma, one can deduce that $z(t)$ moves up and to the left in the region $\pi/6 < \arg z(t) \le \pi/2$. For if $\pi/6 < \arg z(t) \le \pi/2$, then $\pi < \arg[iz(t)^3] \le 2\pi$ and so $0 < \arg[\lambda - iz(t)^3] < \pi$. Thus $\pi/2 < \arg z'(t) < \pi$.

Proof. Since $\operatorname{Im} Z$ is constant on the horizontal line segments,

$$\operatorname{Im} \int_{t_1}^{t_2} \sqrt{\lambda - iz(t)^3} z'(t) dt = \operatorname{Im}\left[Z(z_2) - Z(z_1)\right] = 0 \quad \text{for all } t_1 \text{ and } t_2.$$

And since $\operatorname{Re} Z(z(t))$ is increasing, we similarly deduce that $\sqrt{\lambda - iz(t)^3} z'(t) > 0$. (Notice $\lambda - iz^3 \neq 0$ in $\mathcal{D}$.)

Therefore, $\arg z'(t) = \pi - 1/2 \arg[\lambda - iz(t)^3]$. $\square$

Now we are ready to transform the solution u into the Z-plane. Let U be defined by

$$U(Z(z)) = [\lambda - iz^3]^{1/4} u(z), \quad z \in \mathcal{D}. \tag{9}$$

Since the map $z \mapsto Z$ is one-to-one on $\mathcal{D}$, the transformed solution U is well-defined on $Z(\mathcal{D})$ in the Z-plane. By straightforward computations, one sees that (1) becomes

$$\frac{d^2}{dZ^2}U(Z) + (1 - F(Z))U(Z) = 0, \quad \text{where} \tag{10}$$

$$F(Z(z)) = \frac{1}{4}\frac{(\lambda - iz^3)''}{(\lambda - iz^3)^2} - \frac{5}{16}\frac{((\lambda - iz^3)')^2}{(\lambda - iz^3)^3} = \frac{3z(8\lambda + 7iz^3)}{16i(\lambda - iz^3)^3}. \tag{11}$$

In the next lemma, we provide a series expression for $Z(z)$ near infinity. Using this, in the proof of Proposition 6 we will derive an asymptotic expression for $F(Z)$ in terms of Z near infinity.

Lemma 5. *Fix the base point $z^* \in \mathcal{D}$ with $|z^*|^3 > |\lambda|$. With suitable choice of the integration constant c, we then have a series expression for $Z(z)$ near infinity in $\mathcal{D}$ as follows:*

$$Z(z) = \sqrt{-i}z^{5/2}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\left(\frac{\lambda i}{z^3}\right)^k = \frac{2}{5}\sqrt{-i}z^{5/2}(1 + O(z^{-3})), \tag{12}$$

when $|z|^3 \geq |\lambda|$, $z \in \mathcal{D}$, with

$$\arg\sqrt{-i} = 3\pi/4 \qquad \textit{and} \qquad \arg z^{5/2} = 5/2 \arg z.$$

Note that choosing a different integration constant simply gives a translation in the Z-plane that corresponds to a translation of the transformed solution U.

Proof. From the definition (5) of $Z(z)$, we have that if $|z|^3 > |\lambda|$ then

$$Z(z) = \int_{z^*}^{z} \sqrt{-i}\xi^{3/2}\sqrt{1 + \frac{\lambda i}{\xi^3}}d\xi + c,$$

where $\arg \xi^{3/2} = 3/2 \arg \xi$ and $\arg\sqrt{1 + \lambda i/\xi^3} \in (-\pi/4, \pi/4)$. Hence

$$\begin{aligned} Z(z) &= \int_{z^*}^{z} \sqrt{-i}\xi^{3/2}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\left(\frac{\lambda i}{\xi^3}\right)^k d\xi + c \quad \text{by the binomial series for the square root} \\ &= \sqrt{-i}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(\lambda i)^k \int_{z^*}^{z} \xi^{3/2-3k}d\xi + c \\ &= \sqrt{-i}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2(\lambda i)^k}{5-6k}z^{5/2-3k}, \quad \text{with the constant } c \text{ chosen suitably,} \end{aligned}$$

proving (12) for $|z|^3 > |\lambda|$.

The series in (12) actually converges uniformly and absolutely for $|z|^3 \geq |\lambda|$, because of the following. Let $a_k = \left|\binom{\frac{1}{2}}{k}\right|$. Then for $k \geq 1$,

$$\frac{a_{k+1}}{a_k} = \frac{k - 1/2}{k+1}.$$

So $ka_k - (k+1)a_{k+1} = a_k/2 > 0$. Thus since $\{ka_k\}$, $k \geq 1$ is a positive decreasing sequence, it is bounded above. This implies that the series on the righthand side of (12) converges uniformly and absolutely for $|z|^3 \geq |\lambda|$, and hence is continuous as a function of $z \in \mathcal{D}$ for

$|z|^3 \geq |\lambda|$. And we know that $Z(z)$ is continuous for $z \in \mathcal{D}$. Thus (12) is valid for $|z|^3 \geq |\lambda|$, $z \in \mathcal{D}$. □

We want to know the shape of $Z(\mathcal{D})$. But it is enough for our later argument that $Z(\mathcal{D})$ contains a simply connected subdomain $\mathcal{G}$ with the following four properties:

(i) if $Z \in \mathcal{G}$, then $Z + s \in \mathcal{G}$ for all $s > 0$;
(ii) $\mathcal{G}$ contains $\{Z \in \mathbb{C} : |Z| > L \text{ and } |\arg Z| < \nu\}$, for some L, $\nu > 0$;
(iii) the closure of $\mathcal{G}$ is contained in $Z(\mathcal{D})$;
(iv) there exists a point X_1 on the positive real axis in $\mathcal{G}$ so that any point in $\mathcal{G}$ can be connected to X_1 by at most two line segments in $\mathcal{G}$, one of which is horizontal.

In Lemma 14, we will prove the existence of such a $\mathcal{G}$. But first we estimate the transformed eigenfunction U, only.

Proposition 6. *Suppose $Z(\mathcal{D})$ contains a set $\mathcal{G}$ with the above properties. Let U be the transformed solution (9) of the original solution u of (1), with* $\operatorname{Im}\lambda > 0$. *Then there exists a solution W_1 of the sine equation $W'' + W = 0$ such that*

$$|U(Z) - W_1(Z)| \leq \left[\max_{t \geq 0} |W_1(Z+t)|\right] \left[\exp\left(\int_0^\infty |F(Z+s)|ds\right) - 1\right], \quad Z \in \mathcal{G}. \tag{13}$$

Proof. Hille [14, §7.4] used his method of "asymptotic integration" to investigate asymptotic behavior of the solutions of the equation $u''(z) + P(z)u(z) = 0$, for a certain class of $P(z)$ including all polynomials. Our proof below follows that of Hille. We provide a detailed proof, though, because of the importance of the estimate (13) for our later work, particularly our work in the "finite" parts of $\mathcal{G}$.

First we find asymptotics of $F(Z)$ near infinity. Note that by (11)

$$\begin{aligned} F(Z(z)) &= \frac{3z(8\lambda + 7iz^3)}{16i(\lambda - iz^3)^3} \\ &= \frac{21}{16i} z^{-5}[1 + O(z^{-3})] \quad \text{for large } z \\ &= -\frac{21}{100} Z^{-2}[1 + O(Z^{-\frac{6}{5}})] \quad \text{for large } z, \end{aligned} \tag{14}$$

where the last step is by $Z = \frac{2\sqrt{-i}}{5} z^{5/2}[1 + O(z^{-3})]$ in Lemma 5, which implies $1/5|z|^{5/2} \leq |Z(z)| \leq 3/5|z|^{5/2}$ for large z, say $|z| \geq C_1$. Also $\mathcal{D}$ contains the closure of $Z^{-1}(\mathcal{G})$ so that $Z(z)$ is bounded on $\overline{Z^{-1}(\mathcal{G})} \cap \{z : |z| \leq C_1\}$. So if $|Z|$ is large, then its inverse image lies in $|z| \geq C_1$ so that $1/5|z|^{5/2} \leq |Z(z)| \leq 3/5|z|^{5/2}$ holds. Thus $z \to \infty$ as $Z \to \infty$ and hence $F(Z) = O(|Z|^{-2})$ near infinity in the Z-plane, by (14).

Moreover, $F(Z)$ does not have singularities in $\overline{\mathcal{G}}$. Thus $F(Z)$ is integrable along any half line to infinity in $\mathcal{G}$ and any finite line segment in $\mathcal{G}$. Indeed if Υ is either a half line to infinity or a finite line segment in $\mathcal{G}$, then $\int_\Upsilon |F(Z)||dZ|$ is uniformly bounded, say by M, for all such Υ.

Let W_0 be the solution of the sine equation $W''+W=0$ such that $W_0(X_1)=U(X_1)$ and $W_0'(X_1)=U'(X_1)$. We now define U_0 by

$$U_0(Z):=W_0(Z)+\int_{X_1}^{Z}\sin(Z-\xi)F(\xi)U(\xi)\,d\xi,\quad Z\in\mathcal{G}$$

where the path of integration stays in the simply connected domain $\mathcal{G}$. Then we see that U_0 is analytic and satisfies (10). Since U also satisfies (10), by the uniqueness of solutions with such initial conditions, we have $U\equiv U_0$, so that

$$U(Z)=W_0(Z)+\int_{X_1}^{Z}\sin(Z-\xi)F(\xi)U(\xi)\,d\xi, \tag{15}$$

which we will use to show that $U(Z)$ is bounded on any horizontal strip.

To this end we first fix $d>0$ and let $Z\in\mathcal{G}$ with $|\operatorname{Im}Z|\le d$. The property (iv) of $\mathcal{G}$ says that we can connect Z and X_1 by at most two line segments in $\mathcal{G}$, one of which is horizontal. We let $\Upsilon(s)$, $0\le s\le t$ be a parameterization of the union of these two line segments, moving from X_1 to Z. Then we see that $|\operatorname{Im}\Upsilon(s)|\le|\operatorname{Im}Z|\le d$. Moreover, we may as well assume that $|\Upsilon'(s)|\equiv 1$. We can now find a positive number $K=K(d)$ such that $|W_0(Z)|\le K$ and $|\sin(Z-\Upsilon(s))|\le K$ for all s, since any solution of the sine equation is a linear combination of e^{iZ} and e^{-iZ}, and hence is bounded on any horizontal strip.

Next, we use a Gronwall inequality to show that $U(Z)$ is bounded on the horizontal strip. From (15) we have that

$$\begin{aligned}|U(Z)| &\le |W_0(Z)|+\int_0^t|\sin(Z-\Upsilon(s))F(\Upsilon(s))U(\Upsilon(s))\Upsilon'(s)|\,ds\\ &\le K+K\int_0^t|F(\Upsilon(s))U(\Upsilon(s))|\,ds. \qquad (16)\end{aligned}$$

Let $f(r)=\int_0^r|F(\Upsilon(s))U(\Upsilon(s))|\,ds$, for $0\le r\le t$. Then

$$\begin{aligned}f'(r) &= |F(\Upsilon(r))U(\Upsilon(r))|\\ &\le |F(\Upsilon(r))|(K+Kf(r))\quad\text{by (16).}\end{aligned}$$

Thus

$$f'(r)-K|F(\Upsilon(r))|f(r)\le K|F(\Upsilon(r))|.$$

Now we multiply by $\exp\left(-K\int_0^r|F(\Upsilon(s))|ds\right)$ and then integrate both sides over $r\in[0,t]$:

$$\begin{aligned}f(t)\exp\left(-K\int_0^t|F(\Upsilon(s))|ds\right) &\le \int_0^t K|F(\Upsilon(r))|\exp\left(-K\int_0^r|F(\Upsilon(s))|ds\right)dr\\ &\le 1-\exp\left(-K\int_0^t|F(\Upsilon(s))|ds\right).\end{aligned}$$

Thus

$$f(t)\le\exp\left(K\int_0^t|F(\Upsilon(s))|ds\right)-1\le\exp(2KM)-1.$$

That is, for every $|\mathrm{Im}\, Z| \le d$ and $\Upsilon(s)$, we get $|U(Z)| \le K + K[\exp(2KM) - 1]$, which depends only on d. Thus $U(Z)$ is uniformly bounded for $|\mathrm{Im}\, Z| \le d$, for any $d > 0$ fixed.

Recall that we are trying to approximate $U(z)$ near infinity. We see from (15) that W_0 quite accurately approximates U near X_1 but probably not near infinity. Thus we introduce another function W_1 which will be a "good" approximation of U near infinity, defined by

$$W_1(Z) = U(Z) - \int_Z^{Z+\infty} \sin(\xi - Z) F(\xi) U(\xi) d\xi, \quad Z \in \mathcal{G}, \tag{18}$$

where the integral takes place over a horizontal half line from Z rightwards to infinity. Then it is easy to see that W_1 is a solution of the sine equation, hence is bounded on any horizontal strip. From (18) we have that with $Z = X + iY$,

$$|U(Z) - W_1(Z)| \le \int_X^\infty |F(s+iY)||U(s+iY)| ds =: g(X). \tag{19}$$

Then

$$\begin{aligned} -g'(X) &= |F(Z)||U(Z)| \\ &\le |F(Z)||U(Z) - W_1(Z)| + |F(Z)||W_1(Z)| \\ &\le |F(Z)| g(X) + |F(Z)||W_1(Z)|. \end{aligned}$$

We use another Gronwall-type argument: multiply the above by $\exp\left(-\int_X^\infty |F(s+iY)| ds\right)$, giving

$$-\frac{d}{dX}\left[g(X) \exp\left(-\int_X^\infty |F(s+iY)| ds\right)\right] \le |F(Z)||W_1(Z)| \exp\left(-\int_X^\infty |F(s+iY)| ds\right).$$

Integrating this from X to ∞, and using $g(\infty) = 0$, we get

$$\begin{aligned} & g(X) \exp\left(-\int_X^\infty |F(s+iY)| ds\right) \\ \le\ & \int_X^\infty |F(t+iY)||W_1(t+iY)| \exp\left(-\int_t^\infty |F(s+iY)| ds\right) dt \\ \le\ & \left[\max_{t \ge X} |W_1(t+iY)|\right] \int_X^\infty |F(t+iY)| \exp\left(-\int_t^\infty |F(s+iY)| ds\right) dt \\ =\ & \left[\max_{t \ge X} |W_1(t+iY)|\right] \left[1 - \exp\left(-\int_X^\infty |F(s+iY)| ds\right)\right]. \end{aligned}$$

Hence,

$$g(X) \le \left[\max_{t \ge X} |W_1(t+iY)|\right] \left[\exp\left(\int_X^\infty |F(s+iY)| ds\right) - 1\right].$$

This along with (19) gives (13). □

We want to know more about $W_1(Z)$; in fact, we will prove that $W_1(Z) = C \sin(Z - \pi/4)$ for some $C \in \mathbb{C}$. To this end, we will use the following results of Hille (contained in [14, §7.4]) on the asymptotic behavior of solutions near infinity.

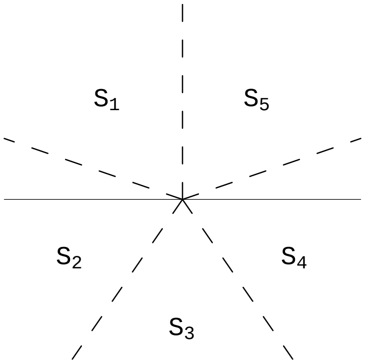

FIGURE 2. The solid line is the real axis, and the dotted rays are the critical rays $\arg z = \theta_j$. In S_2 and S_4, the eigenfunctions decay exponentially near infinity, while in other sectors, the eigenfunctions blow up.

Definition. Define five "critical" rays $\{\arg z = \theta_j\}$ where $\theta_j = (4j - 19)\pi/10, j \in \mathbb{Z}$. Also define *Stokes sectors* $S_j = \{\theta_j < \arg z < \theta_{j+1}\}$ between the critical rays, and $S_j^\epsilon = \{\theta_j + \epsilon < \arg z < \theta_{j+1} - \epsilon\}$ for each $0 < \epsilon < \pi/5$. See Figure 2.

Proposition 7. *Let u be a solution of $-u''(z)+iz^3u(z) = \lambda u(z)$ (with no boundary conditions imposed). Then in each S_j^ϵ,*

$$u(z) \sim (const.)z^{-3/4} \exp\left[\pm\frac{2}{5}i\sqrt{-i}z^{5/2}\left(1 + O(z^{-3})\right)\right] \text{ as } z \to \infty. \tag{21}$$

In particular, $u(z) \to 0$ or ∞ as $z \to \infty$ in S_j^ϵ for each j, ϵ.

If $u(z) \to 0$ as $z \to \infty$ in some S_j^ϵ, then $u(z) \to \infty$ as $z \to \infty$ in S_{j-1}^ϵ and in S_{j+1}^ϵ, and u has a single asymptotic expression in these three Stokes sectors. That is, the asymptotic expression (21) is valid as $z \to \infty$ in $\theta_{j-1} + \epsilon < \arg z < \theta_{j+2} - \epsilon$. Moreover, u has at most finitely many zeros in $\theta_{j-1} + \epsilon < \arg z < \theta_{j+2} - \epsilon$. If $u(z) \to \infty$ in both S_{j-1}^ϵ and S_j^ϵ, then u has infinitely many zeros in $\theta_j - \epsilon < \arg z < \theta_j + \epsilon$, for each $\epsilon > 0$.

Outline of proof of Proposition 7. In this proof, we no longer restrict the map $z \mapsto Z$ on $\mathcal{D}$, but consider it on a neighborhood of infinity.

Deriving (21) is same as proving Proposition 6. Hille transforms the equation (1) into the new equation (10). Then like in the proof of Proposition 6, Hille finds that the solution of (10) is asymptotic to a solution of the sine equation $W''(Z) + W(Z) = 0$. Finally, Hille transforms $W(Z)$ back to the original z-plane, and this gives the asymptotic expression (21) of the solution of (1). That is,

$$u(z) = [\lambda - iz^3]^{-1/4}U(Z(z)) \sim (const.)z^{-3/4}\exp[\pm iZ(z)] \quad \text{as} \quad z \to \infty,$$

where $Z(z) = \frac{2}{5}i\sqrt{-i}z^{5/2}\left[1 + O(z^{-3})\right]$ as $z \to \infty$. One also sees that the asymptotic expression is valid for two consecutive Stokes sectors because a neighborhood of infinity in two consecutive Stokes sectors approximately maps to a neighborhood of infinity in the Z-plane.

Next we suppose that $u \to 0$ as $z \to \infty$ in some S_j^ϵ. Without loss of generality, we may assume that $Z(S_j^\epsilon)$ is approximately the upper half Z-plane and hence the transformed solution is asymptotic to e^{iZ} on $Z(S_j^\epsilon)$. This implies that in (18), $\sin(\xi - Z)U(\xi)$ is bounded on $Z(S_j^\epsilon)$, so that one can change the path of the integration from $Z \to Z + \infty$ to $Z \to Z + i\infty$, and to $Z \to Z - \infty$. That is, we get the asymptotic expression on $-\pi \leq \arg Z \leq 2\pi$ near infinity. (Note that Hille uses the Riemann surface of $\log Z$ as the codomain of the map $z \mapsto Z$, instead of the complex Z-plane, so that the region $-\pi \leq \arg Z \leq 2\pi$ is not overlapped.) This gives an asymptotic expression for three consecutive Stokes sectors.

Lastly, if u is asymptotic to $ce^{\pm iZ(z)}$ in three consecutive Stokes sectors, then since $e^{\pm iZ(z)}$ has no zero and since u and $ce^{\pm iZ(z)}$ are very "close" near infinity, one can prove that u does not have zeros in a neighborhood of infinity. Then since u is analytic, u can have only finitely many zeros in a bounded region. Thus u has at most finitely many zeros in $\theta_{j-1} + \epsilon < \arg z < \theta_{j+2} - \epsilon$. Similarly, if u is asymptotic to $c \sin(\pm Z(z) - Z_0)$, then since $\sin(\pm Z(z) - Z_0)$ has infinitely many zeros near infinity, so does u. □

Now we will show that $W_1(Z) = C \sin(Z - \pi/4)$.

Theorem 8. *The function $W_1(Z)$ in Proposition 6 equals $C \sin(Z - \pi/4)$ for some $C \in \mathbb{C}$, and so*

(23)
$$|U(Z) - C\sin(Z - \pi/4)| \leq (|C| \cosh Y) \left[\exp\left(\int_0^\infty |F(Z+s)| ds\right) - 1\right], \quad Z \in \mathcal{G}.$$

Proof. From Proposition 7, we can deduce the following: The condition that the eigenfunction u satisfies the boundary conditions $u(\pm\infty + 0i) = 0$ implies that $u(z) \to 0$ as $z \to \infty$ in $S_2^\epsilon \cup S_4^\epsilon$, and hence $u(z) \to \infty$ as $z \to \infty$ in $S_1^\epsilon \cup S_3^\epsilon \cup S_5^\epsilon$. Moreover, in $S_1^\epsilon \cup S_2^\epsilon \cup S_3^\epsilon$ and in $S_3^\epsilon \cup S_4^\epsilon \cup S_5^\epsilon$, the asymptotic expressions remain the same, respectively. In S_3^ϵ, which belongs to both cases, the two asymtotic expressions are in fact the same, as we now show.

Note that $\pm(2/5)i\sqrt{-i}z^{5/2}$ lies in the left or right half plane when z lies in S_3^ϵ, but it must be the right half plane by letting $z \to \infty$ in (21), since u blows up in S_3^ϵ. Thus the "+" sign holds. Then the constants in the two different asymptotic expressions must also be the same. Thus (21) becomes

$$u(z) \sim cz^{-3/4} \exp\left[\frac{2}{5} i\sqrt{-i}z^{5/2}\left(1 + O(z^{-3})\right)\right] \text{ as } z \to \infty,$$

for $-3\pi/2 + \epsilon < \arg z < \pi/2 - \epsilon$. Then rotating z through an angle 2π, one sees that

$$u(z) \sim -icz^{-3/4} \exp\left[-\frac{2}{5} i\sqrt{-i}z^{5/2}\left(1 + O(z^{-3})\right)\right] \text{ as } z \to \infty,$$

for $\pi/2 + \epsilon < \arg z < 5\pi/2 - \epsilon$. Thus

$$\begin{aligned} u(z) &\sim +cz^{-3/4} \exp\left[+iZ(z)\right] \quad \text{as} \quad z \to \infty, \quad \text{with} \quad -3\pi/2 + \epsilon < \arg z < \pi/2 - \epsilon, \\ u(z) &\sim -icz^{-3/4} \exp\left[-iZ(z)\right] \quad \text{as} \quad z \to \infty, \quad \text{with} \quad \pi/2 + \epsilon < \arg z < 5\pi/2 - \epsilon, \end{aligned}$$

where $Z(z)$ is defined for $|z|^3 \geq |\lambda|$ by (12). Notice from (12) that $z \mapsto Z$ roughly maps S_1, S_3, S_5 onto the lower half Z-plane and maps S_2, S_4, S_6 onto the upper half Z-plane. Hence we can add the above two asymptotic expressions to yield

$$\begin{aligned} (26) \quad u(z) &\sim cz^{-3/4}\exp[iZ(z)] - icz^{-3/4}\exp[-iZ(z)] \quad \text{as} \quad z \to \infty, \quad z \in S_5^\epsilon \cup S_6^\epsilon, \\ &= 2ce^{3i\pi/4}z^{-3/4}\sin(Z(z) - \pi/4), \end{aligned}$$

where we see that the first term blows up in S_5^ϵ while the second term decays, and vice versa in S_6^ϵ.

So in terms of the transformed solution U, we have that

$$U(Z) \sim C\sin(Z - \pi/4), \quad \text{as} \quad Z \to \infty,\, Z \in \mathcal{G},$$

where $|C| = 2|c|$.

We then show that $W_1(Z) = C\sin(Z - \pi/4)$. Certainly both $W_1(Z)$ and $C\sin(Z - \pi/4)$ are solutions of the sine equation and asymptotic to $U(Z)$, and then the difference of these two, say $C_1e^{iZ} + C_2e^{-iZ}$ for some C_1, $C_2 \in \mathbb{C}$, is also a solution of the sine equation and belongs to the class $o(\sin(Z - \pi/4)$ near infinity. Thus $C_1 = C_2 = 0$ and so $W_1(Z) = C\sin(Z - \pi/4)$. Estimate (23) follows by evaluating the maximum in (13). This completes the proof. □

The following is an application of Rouche's theorem to our problem on the rectangle $O_n = \{Z \in \mathbb{C} : |X - (n\pi + \pi/4)| < \delta_1, |Y| < \delta_2\}$ centered at the zeros of $\sin(Z - \pi/4)$. Here $0 < \delta_1, \delta_2 \leq \pi/2$.

Theorem 9. *If*

$$(27) \qquad \exp\left(\int_0^\infty |F(Z+s)|ds\right) < 1 + \frac{\min\{\sin\delta_1, \sinh\delta_2\}}{\cosh\delta_2} \quad \textit{for all} \quad Z \in \partial O_n,$$

then U has one and only one zero in O_n.

Hille chooses $\delta_1 = \delta_2$ for simplicity. However, we will choose $\delta_1 \neq \delta_2$ to obtain a better estimate on the locations of the zeros.

Proof. Let

$$(28) \qquad V(Z) = U(Z) - C\sin(Z - \pi/4).$$

Then by Proposition 6 and Theorem 8,

$$(29) \qquad |V(Z)| \leq |C||\cosh Y\left[\exp\left(\int_0^\infty |F(Z+s)|ds\right) - 1\right].$$

Thus if for all $Z = X + iY \in \partial O_n$

$$(30) \qquad |C|\cosh Y\left[\exp\left(\int_0^\infty |F(Z+s)|ds\right) - 1\right] < |C\sin(Z - \pi/4)|,$$

then by Rouche's theorem (see, for example, [17, Theorem 8.19]), the number of zeros of U in O_n is the same as that of $C\sin(Z - \pi/4)$, which is exactly one. So we aim to prove that (27) implies (30).

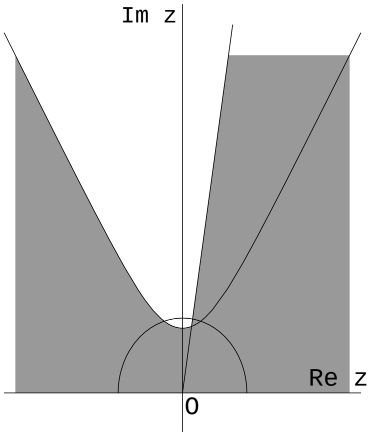

FIGURE 3. The shaded region is the zero-free region of the eigenfunction and its first derivative in the upper half plane. The hyperbola-like curve is $\mathrm{Re}\,(iz^3 - \lambda) = 0$ in $\mathrm{Im}\, z \geq 0$. The ray in the first quadrant is $\{\arg z = 5\pi/12 + \phi/3\}$. The semi-circle is centered at the origin with radius $|\lambda|^{1/3}$.

It is easy to see that $|\sin(Z - \pi/4)| = [\sin^2(X - \pi/4) + \sinh^2 Y]^{1/2}$ has minimum value $\sinh \delta_2$ on the line $|\mathrm{Im}\, Z| = \delta_2$, and minimum value $\sin \delta_1$ on the line $|\mathrm{Re}\,(Z - \pi/4 - n\pi)| = \delta_1$. So

$$\min\{|\sin(Z - \pi/4)| : Z \in \partial O_n\} = \min\{\sin \delta_1, \sinh \delta_2\}.$$

Thus certainly (30) holds if

$$|C| \cosh \delta_2 \left[\exp \left(\int_0^\infty |F(Z+s)| ds \right) - 1 \right] < |C| \min\{\sin \delta_1, \sinh \delta_2\}, \tag{31}$$

which follows from (27). □

With the help of Theorem 9, we will locate some finite zeros of u in the proofs of Theorems 1 and 2. On the other hand, we have the following results on the zero-free region which we will also use in those proofs.

Proposition 7 asserts that there are infinitely many zeros of the eigenfunction u in $\mathrm{Im}\, z \geq 0$ because S_1 and S_5 are blowing up sectors. In [18], I showed that all the zeros of u and u' in $\mathrm{Im}\, z \geq 0$ lie in the region $\mathrm{Re}\,(iz^3 - \lambda) > 0$, which is contained in $\pi/3 < \arg z < 2\pi/3$. In the following theorem, we further extend the zero-free region.

Theorem 10. *Let u be an eigenfunction and let λ be the corresponding eigenvalue with $\phi = \arg \lambda > 0$. That is, $-u''(z) + iz^3 u(z) = \lambda u(z)$ with $u(\pm\infty + 0i) = 0$. Then all the zeros, in $\mathrm{Im}\, z > 0$, of u and u' lie in $5\pi/12 + \phi/3 < \arg z < 2\pi/3$ with $\mathrm{Re}\,(iz^3 - \lambda) > 0$. See the unshaded region in Figure 3.*

In fact, one can even show that if $\phi > 0$, then u and u' do not have zeros in $\pi/2 \leq \arg z < 2\pi/3$. However, we are not going to prove this since the above theorem is enough for our argument later. Figure 3 displays for the case $\arg \lambda > \pi/8$ in which the intersection point of $\operatorname{Re}(iz^3 - \lambda) = 0$ and $\{\arg z = 5\pi/12 + \phi/3\}$ stays inside the semi-circle. If $\arg\lambda < \pi/8$, then the intersection would lie outside the semi-circle.

Proof. In [18], I showed that $|\arg\lambda| \leq \frac{\pi}{5}$, and we know that if λ is an eigenvalue, then so is $\overline{\lambda}$. By [18, Theorem 5], one sees that all the zeros of u and u' in the upper half plane lie in the region $\operatorname{Re}(iz^3 - \lambda) > 0$.

Suppose that $0 < \phi = \arg\lambda \leq \pi/5$. We multiply the differential equation (1) by $\overline{u}$ and integrate the resulting equation along $\{re^{i\theta} : r \leq 0\}$ for a fixed $-\pi/10 < \theta < 3\pi/10$. Proposition 7 gives the needed integrability since the solution is exponentially decaying to zero in S_2. Then we have that if $-\pi/10 < \theta < 3\pi/10$,

$$e^{-4i\theta} u'\overline{u}\Big|_{-\infty}^{0} = e^{-5i\theta}\int_{-\infty}^{0} |u'|^2 dr - e^{-3i\theta}\lambda \int_{-\infty}^{0} |u|^2 dr. \tag{32}$$

Set $\theta = 0$ and take the imaginary part of (32). Then since $\operatorname{Im}\lambda > 0$, we get $\operatorname{Im} u'(0)\overline{u(0)} < 0$. So $\pi < \arg u'(0)\overline{u(0)} < 2\pi$.

We will divide the proof into two cases. First, we suppose that $3\pi/2 \leq \arg u'(0)\overline{u(0)} < 2\pi$. Like we derived (32), we multiply the equation by $\overline{u}$ and integrate it over $[0, iy]$ for $y > 0$. Then we have that

$$\operatorname{Re} u'\overline{u}\big|_0^{iy} = \operatorname{Im}\lambda \int_0^y |u(it)|^2\, dt.$$

Since $\operatorname{Im}\lambda > 0$, we see that $\operatorname{Re} u'(iy)\overline{u(iy)}$ is an increasing function of y. Hence, we have that $\operatorname{Re} u'(iy)\overline{u(iy)} > \operatorname{Re} u'(0)\overline{u(0)} \geq 0$ for all $y > 0$.

Now we let $z = x + iy$ be a point in the first quadrant with $\operatorname{Re}(iz^3 - \lambda) > 0$. Since $\operatorname{Re}(iz^3 - \lambda) = y^3 - 3x^2y - \operatorname{Re}\lambda > 0$, we see that $y^3 - 3t^2y - \operatorname{Re}\lambda > 0$ for all $0 \leq t \leq x$ since $y > 0$. Thus like above we have that

$$\operatorname{Re} u'(t+iy)\overline{u(t+iy)}\Big|_0^x = \int_0^x |u'(t+iy)|^2\, dt + \int_0^x \operatorname{Re}\left(i(t+iy)^3 - \lambda\right)|u(t+iy)|^2\, dt.$$

Since the right hand side is positive, we see that $\operatorname{Re} u'(x+iy)\overline{u(x+iy)} \geq \operatorname{Re} u'(iy)\overline{u(iy)} > 0$. Thus we conclude that u and u' have no zero in the first quadrant; otherwise, we would have $\operatorname{Re} u'(x+iy)\overline{u(x+iy)} = 0$ at the zero in this region. Thus the theorem holds for this case.

Second, we suppose that

$$\pi < \arg u'(0)\overline{u(0)} < 3\pi/2. \tag{33}$$

In (32), since $0 < \phi = \arg\lambda \leq \pi/5$ and $-\pi/10 < \theta < 3\pi/10$, we can choose θ so that $\phi - 3\theta = -\pi/2$. That is, $\theta = \pi/6 + \phi/3$. Now (32) with $\theta = \pi/6 + \phi/3$ becomes

$$\operatorname{Re} e^{-4i(\pi/6+\phi/3)} u'(0)\overline{u(0)} = \cos(5\pi/6 + 5\phi/3)\int_{-\infty}^{0} |u'|^2 dr < 0,$$

since $5\pi/6 < 5\pi/6 + 5\phi/3 < 3\pi/2$. Hence $\operatorname{Re} e^{-4i(\pi/6+\phi/3)}u'(0)\overline{u(0)} < 0$. Then by (33), we have $-4\pi/6 - 4\phi/3 + \arg u'(0)\overline{u(0)} > \pi/2$. That is,

$$\arg u'(0)\overline{u(0)} > \pi + \pi/6 + 4\phi/3. \tag{35}$$

Since $4\pi/3 < 4\theta_0 < 2\pi$ and since $\pi + \pi/6 + 4\phi/3 < \arg u'(0)\overline{u(0)} < 3\pi/2$, we have that $\arg u'(0)\overline{u(0)} < 4\theta_0 - \pi/2$.

Thus this along with (35) shows that $\theta_0 > 5\pi/12 + \phi/3$. Therefore, whether $\operatorname{Re} u'(0)\overline{u(0)}$ is negative or not, we have the desired zero-free region of the eigenfunction and its first derivative. This completes the proof. ☐

3. ESTIMATING THE INVERSE IMAGE OF $|\operatorname{Im} Z| \le \operatorname{arcsinh}(1)$

We first estimate the image under $Z(z)$ of the ray $\{\arg z = \pi/6 + \phi/3\}$.

Lemma 11.

$$\arg Z(\rho|\lambda|^{1/3}e^{i(\pi/6+\phi/3)}) = \begin{cases} 7\pi/6 + 5\phi/6 & \textit{if} \quad \Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{1}{\rho^{3k}} > 0 \\ \pi/6 + 5\phi/6 & \textit{if} \quad \Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{1}{\rho^{3k}} < 0. \end{cases}$$

And

$$|Z(\rho|\lambda|^{1/3}e^{i(\pi/6+\phi/3)})| \ge \begin{cases} |\lambda|^{5/6}\rho^{5/2}|2/5 - 1/\rho^3| & \textit{if} \quad 2/5 - 1/\rho^3 > 0 \\ |\lambda|^{5/6}\rho^{5/2}|2/5 - 1/\rho^3 + 1/28\rho^6| & \textit{if} \quad 2/5 - 1/\rho^3 + 1/28\rho^6 < 0. \end{cases}$$

Moreover, $\operatorname{Im} Z(\rho|\lambda|^{1/3}e^{i(\pi/6+\phi/3)})$ *is a decreasing function of* $\rho \ge 1$.

Proof. Let $z = \rho|\lambda|^{1/3}e^{i(\pi/6+\phi/3)}$. If $\rho \ge 1$, then by (12),

$$Z(\rho|\lambda|^{1/3}e^{i(\pi/6+\phi/3)}) = \sqrt{-i}z^{5/2}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{1}{\rho^{3k}},$$

where $\binom{\frac{1}{2}}{k}\frac{2}{5-6k}$ is an alternating sequence, with absolute values decreasing except the first term $2/5$, so the series converges for all $\rho \ge 1$. Thus the statements of this lemma except the last one are consequences of the properties of alternating series.

In order to prove the last statement of this lemma, we first recall that the map $z \mapsto Z$ is one-to-one on $\{\arg z = \pi/6 + \phi/3, \rho \ge 1\}$ and its entire image lies on the line $Y = X\tan(\pi/6 + 5\phi/6)$ in the Z-plane. $\operatorname{Im} Z(\rho|\lambda|^{1/3}e^{i(\pi/6+\phi/3)})$ tends to the negative infinity as ρ approaches to the positive infinity. Therefore $\operatorname{Im} Z(\rho|\lambda|^{1/3}e^{i(\pi/6+\phi/3)})$ must be a decreasing function of $\rho \ge 1$. ☐

We now estimate the image under $Z(z)$ of the ray $\{\arg z = \pi/2 + \phi/3\}$.

Lemma 12. *If* $\rho > 1$ *then* $\arg Z(\rho|\lambda|^{1/3}e^{i(\pi/2+\phi/3)}) = 5\phi/6$, *and* $|Z(\rho|\lambda|^{1/3}e^{i(\pi/2+\phi/3)})| \ge |\lambda|^{5/6}\rho^{5/2}|2/5 + 1/\rho^3 + 1/28\rho^6|$.

Proof. If $\rho > 1$ then by (12),

$$Z(\rho|\lambda|^{1/3}e^{i(\pi/2+\phi/3)}) = \sqrt{-i}z^{5/2}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{(-1)^k}{\rho^{3k}},$$

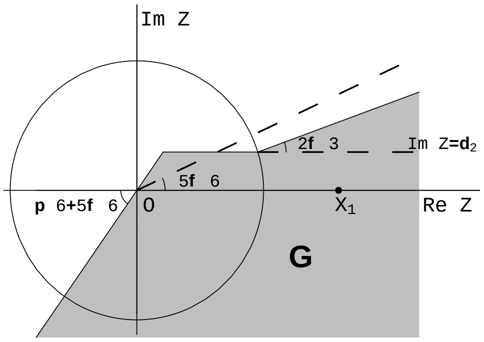

FIGURE 4. The shaded region is $\mathcal{G} \subset Z(\mathcal{D})$. The radius of the circle is $\left|4|\lambda|^{5/6} + i\delta_2\right|$. The half-line from third to first quadrant is a part of the image of $\{\arg z = \pi/6 + \phi/3\}$, the other parts of this figure is explained in the proof of Lemma 14.

where $\binom{\frac{1}{2}}{k}\frac{2(-1)^k}{5-6k}$ is a positive sequence.

Thus $\arg Z(\rho|\lambda|^{1/3}e^{i(\pi/2+\phi/3)}) = \arg(\sqrt{-i}z^{5/2}) = 5\phi/6$. □

Near infinity, $Z(z)$ maps the positive imaginary axis to near the positive real axis.

Lemma 13. *If* $\rho \geq 2^{1/3}$ *then* $\operatorname{Im} Z(\rho|\lambda|^{1/3}i) \geq |\lambda|^{5/6}\sin\phi/\rho^{1/2}$.

Proof. By (12), if $\rho \geq 1$, then

$$\begin{aligned}
&\operatorname{Im} Z(\rho|\lambda|^{1/3}i)\\
=\;& |\lambda|^{5/6}\rho^{5/2}\operatorname{Im}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{(-e^{i\phi})^k}{\rho^{3k}}\\
=\;& |\lambda|^{5/6}\rho^{5/2}\Sigma_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{(-1)^k\sin k\phi}{\rho^{3k}} \qquad (36)\\
\geq\;& |\lambda|^{5/6}\rho^{5/2}\left[\Sigma_{k=1}^{\lfloor\pi/\phi\rfloor}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{(-1)^k\sin k\phi}{\rho^{3k}} - \Sigma_{k=\lfloor\pi/\phi\rfloor+1}^{\infty}\left|\binom{\frac{1}{2}}{k}\frac{2}{(5-6k)\rho^{3k}}\right|\right]\\
\geq\;& |\lambda|^{5/6}\rho^{5/2}\frac{\sin\phi}{\rho^3}, \quad \text{if } \rho \geq 2^{1/3},
\end{aligned}$$

where $\lfloor\pi/\phi\rfloor$ is the largest integer $\leq \pi/\phi$. □

We are now ready to show the existence of the $\mathcal{G}$ that we used in Proposition 6.

Lemma 14. *Let* $\mathcal{G}$ *be the union of* $\{Z = X + iY : Y \leq \delta_2$ *and* $Y \leq X\tan(\pi/6 + 5\phi/6)\}$ *and* $\{Z = X + iY : 0 \leq Y \leq \delta_2 + (X - 4|\lambda|^{5/6})\tan(2\phi/3)\}$. *If* λ *is sufficiently large, namely* $|\lambda| \geq \max\left\{4, 2\left(\frac{\delta_2}{\sin\phi}\right)^{6/5}\right\}$, *then* $\mathcal{G} \subset Z(\mathcal{D})$.

Proof. By Lemma 11, $\operatorname{Im} Z(\rho|\lambda|^{1/3}e^{i(\pi/6+\phi/3)})$ is continuous and decreasing, and it lies on the line $Y = X\tan(\pi/6 + 5\phi/6)$ for all $\rho \geq 1$. Also, if $|\lambda| \geq 4$ then

$$\begin{aligned}
\operatorname{Im} Z(|\lambda|^{1/3}e^{i(\pi/6+\phi/3)}) \;&\geq\; |\lambda|^{5/6}\frac{79}{140}\sin(\frac{\pi}{6} + \frac{5\phi}{6}) \quad \text{by Lemma 11}\\
&\geq\; \frac{79}{70\cdot 2^{1/3}} > 0.89 > \operatorname{arcsinh}(1).
\end{aligned}$$

Let $z(t)$ be the inverse image under $Z(z)$ of the horizontal line $\operatorname{Im} Z = \delta_2$ with $\operatorname{Re} Z(z(t))$ increasing where $0 < \delta_2 \leq \operatorname{arcsinh}(1)$. Then $z(t)$ passes through $\{\arg z = \pi/6 + \phi/3\}$ at $|z| \geq |\lambda|^{1/3}$. From (36), it is not difficult to see that $\operatorname{Im} Z(\rho|\lambda|^{1/3} i) \to 0$ as $\rho \to \infty$. Since $\operatorname{Im} Z(z(t)) = \delta_2 > 0$, $z(t)$ must hit the positive imaginary axis in the z-plane. Note that from Lemma 4, $z(t)$ moves up to the left.

Now by Lemma 13,

$$\operatorname{Im} Z(2^{1/3}|\lambda|^{1/3} i) \quad \geq \quad |\lambda|^{5/6} \frac{\sin\phi}{2^{1/6}}.$$

Thus if $|\lambda| \geq 2\left(\delta_2/\sin\phi\right)^{6/5}$, then $z(t)$ must hit the positive imaginary axis at $|z| \geq (2|\lambda|)^{1/3}$.

If $|z| = \rho|\lambda|^{1/3}$ with $\rho \geq 1$, then from (12),

$$|Z(z)| \quad \leq \quad \rho^{5/2}|\lambda|^{5/6} \Sigma_{k=0}^{\infty} \left| \binom{\frac{1}{2}}{k} \frac{2}{5-6k} \frac{1}{\rho^{3k}} \right|.$$

Thus, if $\rho = 2^{1/3}$, then $|Z(z)| \leq 4|\lambda|^{5/6}$.

The one-to-one mapping $z \mapsto Z$ maps a simply connected domain onto a simply connected domain, and its boundary onto the corresponding boundary. Thus, by Lemmas 11 and 12, $Z(\mathcal{D})$ contains $\{Z \in \mathbb{C} : -5\pi/6 + 5\phi/6 \leq \arg Z < 5\phi/6, |Z| \geq 4|\lambda|^{5/6}\}$ if $|\lambda| \geq 4$ and $2\left(\delta_2/\sin\phi\right)^{6/5}$.

Consider the simply connected domain in $\mathcal{D}$ that contains the positive imaginary axis near infinity and is enclosed by the union of $\{\arg z = \pi/6 + \phi/3\}$, $\{z(t)\}$, $\{\arg z = \pi/2 + \phi/3\}$ and an arc of the circle centered at the origin with radius $(2|\lambda|)^{1/3}$ to connect $z(t)$ and $\{\arg z = \pi/2 + \phi/3\}$. The image under $Z(z)$ of this set certainly contains the desired set $\mathcal{G}$. This completes the proof. $\square$

The ray $\{\arg z = 5\pi/12 + \phi/3\}$ appears in the description of the zero-free region of the eigenfunction. The following estimation of the image of this ray will be used in the proofs of Theorems 1 and 2.

Lemma 15. *Let*

$$\begin{aligned} K_1(\rho) &= \left(\frac{2}{5} + \frac{1}{\sqrt{2}\rho^3}\right) \\ &\quad + i\left(\frac{1}{\sqrt{2}\rho^3} + \frac{1}{28\rho^6} + \frac{1}{104\sqrt{2}\rho^9} - \frac{7}{3200\sqrt{2}\rho^{15}} - \frac{21}{15872\rho^{18}} - \frac{33}{37888\sqrt{2}\rho^{21}}\right), \\ K_2(\rho) &= \left(\frac{2}{5} + \frac{1}{\sqrt{2}\rho^3} - \frac{1}{104\sqrt{2}\rho^9} - \frac{5}{1216\rho^{12}} - \frac{7}{3200\sqrt{2}\rho^{15}}\right) + i\left(\frac{1}{\sqrt{2}\rho^3} + \frac{1}{28\rho^6} + \frac{1}{104\sqrt{2}\rho^9}\right). \end{aligned}$$

Also let

$$L_1(\rho) = [(\operatorname{Re} K_2(\rho))^2 + (\operatorname{Im} K_1(\rho))^2]^{1/2}.$$

Then

$$\frac{5\phi}{6} - \frac{5\pi}{24} + \arg K_1(\rho) \leq \arg Z(\rho|\lambda|^{1/3} e^{i(5\pi/12+\phi/3)}) \leq \frac{5\phi}{6} - \frac{5\pi}{24} + \arg K_2(\rho), \qquad \textit{for all } \rho \geq 1.$$

And

$$L_1(\rho)|\lambda|^{5/6}\rho^{5/2} \leq |Z(\rho|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)})|, \qquad \text{for all } \rho \geq 1. \tag{37}$$

Moreover, $\rho^{5/2}L_1(\rho)$ *is an increasing function of* $\rho \geq 1$ *and* $\arg K_2(\rho)$ *is a decreasing function of* $\rho \geq 1$.

Proof. Let $z = \rho|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)}$. Then from (12), we have

$$Z(\rho|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)}) = |\lambda|^{5/6}\rho^{5/2}e^{i(5\phi/6-5\pi/24)}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{e^{-3k\pi i/4}}{\rho^{3k}}.$$

Let $A(\rho) = \Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2}{5-6k}\frac{e^{-3k\pi i/4}}{\rho^{3k}}$. Note here that

$$\begin{aligned} \operatorname{Re} A(\rho) &= \frac{2}{5} + \frac{1}{\sqrt{2}\rho^3} - \frac{1}{104\sqrt{2}\rho^9} - \frac{5}{1216\rho^{12}} - \frac{7}{3200\sqrt{2}\rho^{15}} + \frac{33}{37888\sqrt{2}\rho^{21}} + \cdots, \\ \operatorname{Im} A(\rho) &= \frac{1}{\sqrt{2}\rho^3} + \frac{1}{28\rho^6} + \frac{1}{104\sqrt{2}\rho^9} - \frac{7}{3200\sqrt{2}\rho^{15}} - \frac{21}{15872\rho^{18}} - \frac{33}{37888\sqrt{2}\rho^{21}} + \cdots. \end{aligned}$$

It is easy to see that $\operatorname{Re} A$ and $\operatorname{Im} A$ are positive for all $\rho \geq 1$. Also,

$$0 < \operatorname{Re} K_2 \leq \operatorname{Re} A \leq \operatorname{Re} K_1 \quad \text{and} \quad 0 < \operatorname{Im} K_1 \leq \operatorname{Im} A \leq \operatorname{Im} K_2$$

for all $\rho \geq 1$. Thus, these inequalities prove the lemma except the last statement.

By a straight forward computation of the derivative of $(\rho^{5/2}L_1)^2$, one can show that $\rho^{5/2}L_1$ is an increasing function of ρ. Also, one can easily see that

$$\frac{\rho^3 \operatorname{Im} K_2}{\rho^3 \operatorname{Re} K_2} = \left(\frac{1}{\sqrt{2}} + \frac{1}{28\rho^3} + \frac{1}{104\sqrt{2}\rho^6}\right)\Big/\left(\frac{2\rho^3}{5} + \frac{1}{\sqrt{2}} - \frac{1}{104\sqrt{2}\rho^6} - \frac{5}{1216\rho^9} - \frac{7}{3200\sqrt{2}\rho^{12}}\right)$$

is a decreasing function of $\rho \geq 1$ and so is $\arg K_2$. ☐

The next corollary gives a sufficient condition under which $Z^{-1}(O_1)$ entirely lies in the zero-free region of the eigenfunction and its first derivative.

Corollary 16. *If* $|\lambda|^{5/6}L_1 \sin(5\phi/6 - 5\pi/24 + \arg K_1)\big|_{\rho=1} \geq \delta_2$ *and*

$$|\lambda|^{5/6}L_1 \cos(5\phi/6 - 5\pi/24 + \arg K_2)\big|_{\rho=1} \geq \pi + \pi/4 + \delta_1,$$

then $Z^{-1}(O_1)$ *entirely lies in the zero-free region of the eigenfunction and its first derivative.*

Proof. By the last statement of Lemma 15, one can see that if

$$\begin{aligned} \operatorname{Im} Z(|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)}) &> |\lambda|^{5/6}L_1 \sin(5\phi/6 - 5\pi/24 + \arg K_1)\big|_{\rho=1} \\ &\geq \delta_2, \end{aligned}$$

then the inverse image of $|\operatorname{Im} Z| \leq \delta$ intersects $\{\arg z = 5\pi/12 + \phi/3\}$ at $\rho \geq 1$. By (37) and the last statement of Lemma 15, we see that $Z(\rho|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)})$ lies outside the the circle of the radius $|\lambda|^{5/6}L_1(1)$ centered at the origin. Since

$$\operatorname{Re} Z(|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)}) \geq |\lambda|^{5/6}L_1 \cos(5\phi/6 - 5\pi/24 + \arg K_2)\big|_{\rho=1},$$

any horizontal line in the Z-plane starting near the origin and moving to the right intersects the vertical line $\operatorname{Re} Z = |\lambda|^{5/6} L_1 \cos(5\phi/6 - 5\pi/24 + \arg K_2)\big|_{\rho=1}$ before it hits the curve $\big\{Z(\rho|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)}) : \rho \geq 1\big\}$. That is, the inverse image of

$$\{Z \in \mathcal{G} : |\operatorname{Im} Z| \leq \delta_2, \operatorname{Re} Z \leq |\lambda|^{5/6} L_1 \cos(5\phi/6 - 5\pi/24 + \arg K_2)\big|_{\rho=1}\}$$

entirely lies in the zero-free region of the eigenfunction and its first derivative. □

We have the following estimates on the image of $\{\arg z = \pi/3 + \phi/3\}$ under $Z(z)$ that will be used in the proof of Theorem 2.

Lemma 17. *The following three estimates hold, provided that all arguments of the sine function are positive.*

$$\begin{aligned}
\operatorname{Im} Z(|\lambda|^{1/3}e^{i(\pi/3+\phi/3)}) &\geq 1.057|\lambda|^{5/6}\sin(5\phi/6 - 0.094).\\
\operatorname{Im} Z(1.1292|\lambda|^{1/3}e^{i(\pi/3+\phi/3)}) &\geq 1.071|\lambda|^{5/6}\sin(5\phi/6 - 0.245).\\
\operatorname{Im} Z(1.2298|\lambda|^{1/3}e^{i(\pi/3+\phi/3)}) &\geq 1.112|\lambda|^{5/6}\sin(5\phi/6 - 0.368).
\end{aligned}$$

Proof. The proof is almost identical to that of the previous lemma.

Let

$$K_3(\rho) = \left(\frac{2}{5} - \frac{1}{28\rho^6} + \frac{5}{1216\rho^{12}}\right) + i\left(\frac{1}{\rho^3} - \frac{1}{104\rho^9} + \frac{7}{3200\rho^{15}} - \frac{33}{37888\rho^{21}}\right)$$

and

$$K_4(\rho) = \left(\frac{2}{5} - \frac{1}{28\rho^6} + \frac{5}{1216\rho^{12}} - \frac{21}{15872\rho^{18}}\right) + i\left(\frac{1}{\rho^3} - \frac{1}{104\rho^9} + \frac{7}{3200\rho^{15}}\right).$$

Also let $L_2(\rho)$ be

$$\left[(\operatorname{Re} K_4(\rho))^2 + (\operatorname{Im} K_3(\rho))^2\right]^{1/2}.$$

Then for all $\rho \geq 1$ the following hold:

$$5\phi/6 - 5\pi/12 + \arg K_3 < \arg Z(\rho|\lambda|^{1/3}e^{i(\pi/3+\phi/3)}) < 5\phi/6 - 5\pi/12 + \arg K_4, \quad \text{and}$$

$$|\lambda|^{5/6}\rho^{5/2}L_2 < |Z(\rho|\lambda|^{1/3}e^{i(\pi/3+\phi/3)})|.$$

From the above, we get $\operatorname{Im} Z(\rho|\lambda|^{1/3}e^{i(\pi/3+\phi/3)}) \geq L_2|\lambda|^{5/6}\rho^{5/2}\sin(5\phi/6 - 5\pi/12 + \arg K_3)$ if $5\phi/6 - 5\pi/12 + \arg K_3 \geq 0$.

If $\rho = 1$ then

$$\begin{aligned}
&L_2|\lambda|^{5/6}\rho^{5/2}\sin(5\phi/6 - 5\pi/12 + \arg K_3)\big|_{\rho=1}\\
&\geq 1.057|\lambda|^{5/6}\sin(5\phi/6 - 0.094),
\end{aligned}$$

provided $5\phi/6 - 0.094 \geq 0$. Thus we have that if $\phi \geq 0.1128$,

$$\operatorname{Im} Z(|\lambda|^{1/3}e^{i(\pi/3+\phi/3)}) \geq 1.057|\lambda|^{5/6}\sin(5\phi/6 - 0.094).$$

If $\rho = 1.1292$ then

$$L_2|\lambda|^{5/6}\rho^{5/2}\sin(5\phi/6 - 5\pi/12 + \arg K_3)\big|_{\rho=1.1292} \quad \geq \quad 1.071|\lambda|^{5/6}\sin(5\phi/6 - 0.245),$$

provided $5\phi/6 - 0.245 \geq 0$.

If $\rho = 1.2298$ then

$$L_2|\lambda|^{5/6}\rho^{5/2}\sin(5\phi/6 - 5\pi/12 + \arg K_3)\big|_{\rho=1.2298} \quad \geq \quad 1.112|\lambda|^{5/6}\sin(5\phi/6 - 0.368),$$

provided $5\phi/6 - 0.368 \geq 0$. ☐

In order to estimate a lower bound of $|z(t)|$ in Lemma 23, we will use the following.

Lemma 18.

$$\begin{aligned} \operatorname{Im} Z(|\lambda|^{1/3}e^{i(5\pi/18+\phi/3)}) \quad &\geq \quad |\lambda|^{5/6}\frac{\sqrt{9395311}}{3640}\sin(5\phi/6 - 4\pi/9 + \arctan(\frac{535795\sqrt{3}}{119776})) \\ &\geq \quad 0.842|\lambda|^{5/6}\sin(5\phi/6 - 0.047), \end{aligned}$$

provided $5\phi/6 - 0.047 \geq 0$.

Proof. Let $z = |\lambda|^{1/3}e^{i(5\pi/18+\phi/3)}$. Then from (12), we have

$$\begin{aligned} Z(|\lambda|^{1/3}e^{i(5\pi/18+\phi/3)}) \quad &= \quad |\lambda|^{5/6}e^{i(5\phi/6-5\pi/9)}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2e^{-k\pi i/3}}{5-6k} \quad \text{and} \\ \Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2e^{-k\pi i/3}}{5-6k} \quad &= \quad \left(\frac{2}{5} - \frac{1}{2} - \frac{1}{56} + \frac{1}{104} - \frac{5}{2432} - \frac{7}{6400} + \frac{21}{15872} - \cdots\right) \\ &+ \quad i\sqrt{3}\left(\frac{1}{2} - \frac{1}{56} + \frac{5}{2432} - \frac{7}{6400} + \frac{33}{75776} - \cdots\right). \end{aligned}$$

Thus we have that

$$\begin{aligned} \frac{2}{5} - \frac{1}{2} - \frac{1}{56} \leq \operatorname{Re}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2e^{-k\pi i/3}}{5-6k} \quad &\leq \quad \frac{2}{5} - \frac{1}{2} - \frac{1}{56} + \frac{1}{104} < 0 \\ 0 < \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{56} \leq \operatorname{Im}\Sigma_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{2e^{-k\pi i/3}}{5-6k} \quad &\leq \quad \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{56} + \frac{5\sqrt{3}}{2432}. \end{aligned}$$

The lemma is a consequence of these inequalities. ☐

4. ESTIMATING $\int_Z^{Z+\infty}|F(Z+s)|ds$

In this section we will estimate $\int_Z^{Z+\infty}|F(Z+s)|ds$ for $Z \in \mathcal{G}$, in order to locate some zeros of the eigenfunctions near the origin.

The equation (14) gives an asymptotic expression of $F(Z)$ near infinity. However, in a bounded region, $F(Z)$ might not be accurately expressed by (14). Thus, instead of estimating $\int_Z^{Z+\infty}|F(Z+s)|ds$ in the Z-plane over horizontal half lines, we will estimate the integral in the z-plane with the change of variables (5). Advantage is that we know $F(Z(z))$ exactly by (11), while disadvantage is we do not know the exact integral paths which are the inverse images of horizontal half lines.

The main idea of the proofs of Theorems 1 and 2 is that on one hand, we prove that by estimating $\int_Z^{Z+\infty} |F(Z+s)|ds$, there is a zero of U near the origin in the Z-plane for certain λ. On the other hand, we show that the inverse image of this zero must lie in the zero free region given by Theorem 10. This contradiction leads to the main results of this paper. We will use the following lemma in estimating $\int_Z^{Z+\infty} |F(Z+s)|ds$.

Lemma 19. *Let $z(t) \in \mathcal{D}$ be a parameterization of the inverse image of a horizontal half line in $|\operatorname{Im} Z| \le \delta_2 \le \operatorname{arcsinh}(1)$. Suppose $\operatorname{Re} Z(z(t))$ is increasing function and $|z(t)|^3 > |\lambda|$ for $\arg z(t) > \pi/2$.*

If $\frac{d|z|}{dt}(t_0) > 0$ for some t_0, then $\frac{d|z|}{dt}(t) > 0$ for all $t > t_0$; and if $\frac{d|z|}{dt}(t_0) \le 0$ for some t_0, then $\frac{d|z|}{dt}(t) < 0$ for all $t < t_0$

In particular, if $|z(t)|^3 > |\lambda|$ for $\arg z(t) > 5\pi/12 + \phi/3$, then $\frac{d|z|}{dt}(t) > 0$; and if $\arg z(t) \le 3\pi/10$, then $\frac{d|z|}{dt}(t) < 0$.

Proof. Certainly, $\frac{d|z|}{dt}$ and $\frac{d|z|^2}{dt}$ have a same sign. So we will use $\frac{d|z|^2}{dt}$ to prove this lemma.

By a reparameterization of $z(t)$, one can assume that $\frac{d}{dt}Z(z(t)) \equiv 1$. That is,

$$z'(t)\sqrt{\lambda - iz(t)^3} \equiv 1.$$

Then

$$\begin{aligned} \frac{d}{dt}|z(t)|^2 &= 2\operatorname{Re}\frac{\overline{z(t)}}{\sqrt{\lambda - iz(t)^3}} \quad \text{and} \\ \arg\frac{\overline{z(t)}}{\sqrt{\lambda - iz(t)^3}} &= -\arg z(t) - (\pi + 1/2\arg(\lambda - iz(t)^3)). \end{aligned} \tag{38}$$

Since for $z(t) \in \mathcal{D}$, $0 \le \arg(\lambda - iz(t)^3) < \pi + \phi$,

$$-5\pi/2 < \arg\frac{\overline{z(t)}}{\sqrt{\lambda - iz(t)^3}} < -\theta - \pi.$$

Hence, it is clear that

$$\frac{d}{dt}|z(t)|^2 > 0 \quad \text{if and only if} \quad \arg\overline{z(t)}/\sqrt{\lambda - iz(t)^3} < -3\pi/2. \tag{40}$$

When $z(t) \in \mathcal{D}$ and $\arg z(t) \ge \pi/6 + \phi/3$, we see that

$$\arg(\lambda - iz(t)^3) \le \arg(-iz(t)^3) = -\pi/2 + 3\arg z(t).$$

Then, from (38)

$$\arg\overline{z(t)}/\sqrt{\lambda - iz(t)^3} \ge -3\pi/4 - 5/2\arg z(t).$$

So if $\pi/6 + \phi/3 \le \arg z(t) \le 3\pi/10$, then $\frac{d|z|}{dt}(t) \le 0$. If $\pi/6 \le \arg z(t) < \pi/6 + \phi/3$, one can see that $\frac{d|z|}{dt}(t) \le 0$ by a similar argument.

Suppose that $\frac{d|z|^2}{dt}(t_0) > 0$ for some t_0, and suppose $t > t_0$. If $\arg z(t) \ge \pi/2$, then since $\arg(\lambda - iz(t)^3) > 0$, we have that from (38), $\arg\frac{\overline{z(t)}}{\sqrt{\lambda - iz(t)^3}} < -3\pi/2$. Thus $\frac{d|z|^2}{dt}(t) > 0$.

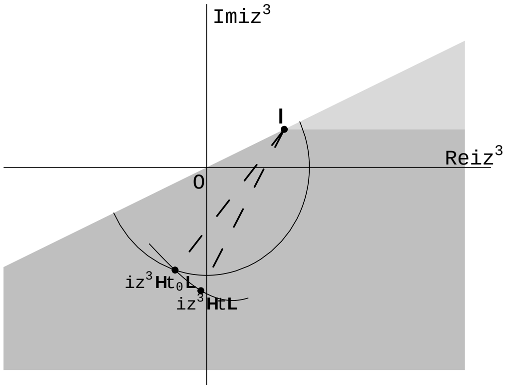

FIGURE 5. This figure shows why (41) holds. If $iz(t)^3$ lies in the darker shaded region, $z(t)$ in the z-plane moves up to the left, while if $iz(t)^3$ lies in the lighter shaded region, $z(t)$ moves up to the right.

If $3\pi/10 < \arg z(t_0) < \pi/2$, then we first recall that in this region $z(t)$ moves up to the left. So $\arg z(t)$ is increasing. Since $t > t_0$ with t close to t_0, by the continuity of $|z(t)|$, we see that $|z(t)|$ is also increasing. So we get that

$$\arg(\lambda - iz(t)^3) \geq (\lambda - iz(t_0)^3). \tag{41}$$

See Figure 5. Thus we see now that even if t is little away from t_0,

$$\arg \overline{z(t)}/\sqrt{\lambda - iz(t)^3} \leq \arg \overline{z(t_0)}/\sqrt{\lambda - iz(t_0)^3} < -3\pi/2.$$

Therefore, if $\frac{d|z|^2}{dt}(t_0) > 0$ for some t_0, then $\frac{d|z|^2}{dt}(t) > 0$ for all $t > t_0$.

A similar argument shows that if $\frac{d|z|^2}{dt}(t_0) = 0$ for some t_0, then $\frac{d|z|^2}{dt}(t) > 0$ for all $t > t_0$. This completes the proof. $\square$

One can get properties similar to the above lemma, regarding $\arg z(t)$.

Lemma 20. *Let $z(t) \in \mathcal{D}$ be a parameterization of the inverse image of a horizontal half line in $|\operatorname{Im} Z| \leq \delta_2 \leq \operatorname{arcsinh}(1)$. Suppose $\operatorname{Re} Z(z(t))$ is increasing function and $|z(t)|^3 > 2|\lambda|$ for $\arg z(t) = \pi/2$. Then $\arg z(t)$ is increasing if $\pi/6 + \phi/3 \leq \arg z(t) \leq \pi/2$, and $\frac{d}{dt}(\arg z(t))$ changes its sign at most once.*

Also, $\arg z(t) \leq \pi/2 + \phi/3$.

Proof. From the paragraph before the proof of Lemma 4, we see that in the first quadrant, $z(t)$ moves up to the left. This implies that $\arg z(t)$ is increasing if $\pi/6+\phi/3 < \arg z(t) \leq \pi/2$ and $\frac{d}{dt}(\arg z(t))$ does not change its sign in this region.

Suppose that $\frac{d}{dt}(\arg z(t))$ changes its sign more than once. Then there exists $\pi/2 \leq \theta \leq \pi/2+\phi/3$ such that $z(t)$ intersects the ray $\{re^{i\theta} : r > 1\}$ at least 3 times, say at $r_1 < r_2 < r_3$. Since $\sqrt{\lambda - iz^3}$ is analytic in $\mathcal{D}$, the following integrals do not depend on the integral paths.

Thus we have that

$$\begin{aligned} 0 &= \operatorname{Im} Z(r_2 e^{i\theta}) - \operatorname{Im} Z(r_1 e^{i\theta}) = \operatorname{Im} \int_{r_1}^{r_2} \sqrt{\lambda - ir^3 e^{3i\theta}}\, e^{i\theta} dr \quad \text{and} \qquad (42)\\ 0 &= \operatorname{Im} Z(r_3 e^{i\theta}) - \operatorname{Im} Z(r_2 e^{i\theta}) = \operatorname{Im} \int_{r_2}^{r_3} \sqrt{\lambda - ir^3 e^{3i\theta}}\, e^{i\theta} dr. \qquad (43)\end{aligned}$$

It is then not difficult to see that

$$\begin{aligned} \arg\left(\sqrt{\lambda - ir^3 e^{3i\theta}}\right) &< \arg\left(\sqrt{\lambda - ir_2^3 e^{3i\theta}}\right) \quad \text{if} \quad r < r_2 \\ \arg\left(\sqrt{\lambda - ir^3 e^{3i\theta}}\right) &> \arg\left(\sqrt{\lambda - ir_2^3 e^{3i\theta}}\right) \quad \text{if} \quad r > r_2. \end{aligned}$$

Thus it is impossible to have that both (42) and (43) hold. Therefore, $\frac{d}{dt}(\arg z(t))$ changes its sign at most once.

Finally, the last statement is clear from Lemma 14. $\square$

Theorem 21. *Let $z(t)$ be a parameterization of the inverse image of a horizontal line in $|\operatorname{Im} Z| \le \delta_2$ with $\operatorname{Re} Z(z(t))$ increasing. Suppose that if $\arg z(t) = \pi/2$ for some t, then $|z(t)| \ge (2|\lambda|)^{1/3}$, and suppose that if $\arg z(t) = \pi/6 + \phi/3$, then $|\lambda|^{1/3} \le |z(t)| \le (3|\lambda|)^{1/3}$. Then we have the following estimate.*

$$\int_Z^{Z+\infty} |F(Z+s)| ds \le \frac{1}{|\lambda|^{5/6}} B(|\arg \lambda|), \qquad (44)$$

where

$$\begin{aligned} B(\phi) &:= \left(\frac{6}{11\cdot 2^{11/6}} + \frac{21}{20\cdot 2^{5/6}} - \frac{3}{11\cdot 3^{11/6}} - \frac{21}{40\cdot 3^{5/6}}\right)\frac{1}{(1-1/2)^{5/2}} \\ &+ \left(\frac{24\cdot 2^{2/3} + 21\cdot 2^{5/3}}{16\cdot 2^{15/6}}\right)\frac{\pi/3+\phi/3}{(1-1/2)^{5/2}} \\ &+ \frac{1}{\sin^{5/2}\phi}\left\{\frac{6}{11}\left(\frac{1}{2^{11/6}} - \frac{1}{\sin^{11/2}(\pi/6+\phi/3)}\right) + \frac{21}{20}\left(\frac{1}{2^{5/6}} - \frac{1}{\sin^{5/2}(\pi/6+\phi/3)}\right)\right. \\ &+ \left.\left(\frac{3}{2\sin^{11/2}(\pi/6+\phi/3)} + \frac{21}{16\sin^{5/2}(\pi/6+\phi/3)}\right)(\pi/3-\phi/3)\right\}. \end{aligned}$$

Proof. Let $z(t)$ be a parameterization of the inverse image Γ of a horizontal half line in $|\operatorname{Im} Z| \le \operatorname{arcsinh}(1)$ such that $\operatorname{Re} Z(z(t))$ is increasing. Then for all $Z \in \mathcal{G}$, (11) with the change of variables $Z = \int^z \sqrt{\lambda - i\xi^3} d\xi$, gives

$$\int_Z^{Z+\infty} |F(Z+s)| ds = \int_\Gamma \left|\frac{3z(8\lambda + 7iz^3)}{16(\lambda - iz^3)^{5/2}}\right| |dz|. \qquad (45)$$

Moreover, $|dz| \le |dr| + r|d\theta|$ where $z = re^{i\theta}$. Now we divide the right hand side into two cases; when $|z(t)| \ge (2|\lambda|)^{1/3}$ and when $|z(t)| \le (2|\lambda|)^{1/3}$.

First, we estimate a part of the integral for $|z(t)| \geq (2|\lambda|)^{1/3}$.

$$\begin{aligned} & \int_{|z(t)|\geq (2|\lambda|)^{1/3}} \left| \frac{3z(8\lambda + 7iz^3)}{16(\lambda - iz^3)^{5/2}} \right| (|dr| + r|d\theta|) \\ \leq & \int_{(3|\lambda|)^{1/3}}^{(2|\lambda|)^{1/3}} \frac{24|\lambda|r + 21r^4}{16r^{15/2}(1-|\lambda|r^{-3})^{5/2}}(-dr) + \int_{(2|\lambda|)^{1/3}}^{\infty} \frac{24|\lambda|r + 21r^4}{16r^{15/2}(1-|\lambda|r^{-3})^{5/2}} dr \\ + & \left. \frac{24|\lambda|r^2 + 21\cdot r^5}{16\cdot r^{15/2}(1-|\lambda|r^{-3})^{5/2}} \right|_{r=(2|\lambda|)^{1/3}} (\pi/2 + \phi/3 - (\pi/6 + \phi/3) + \phi/3), \end{aligned} \tag{46}$$

where $|dr| = dr$ or $-dr$ depending on the sign of $d|z|/dt$ (see Lemma 19)and likewise for $|d\theta|$ (see Lemma 20), and we bound all denominators from below by

$$16|\lambda - iz^3|^{5/2} \geq 16r^{15/2}|1 + \lambda i/z^3|^{5/2} \geq 16r^{15/2}|1 - |\lambda|/r^3|^{5/2}.$$

The first term in (46) is for $z(t)$ near $\{\arg z = \pi/6 + \phi/3\}$ while $|z(t)|$ is decreasing, the second term is for near $\{\arg z = \pi/2\}$ while $|z(t)|$ is increasing The last term is for $d\theta$ that is given by the product of the maximum value of the integrand and an upper bound of $\int_\Gamma |d\theta|$. This upper bound is justified by Lemma 20.

The first two terms are not easy to integrate, so replace r in $(1 - |\lambda|r^{-3})^{5/2}$ by $(2|\lambda|)^{1/3}$ which makes them larger, and integrate the rests. Then we have that

$$\begin{aligned} & \int_{|z(t)|\geq (2|\lambda|)^{1/3}} \left| \frac{3z(8\lambda + 7iz^3)}{16(\lambda - iz^3)^{5/2}} \right| (|dr| + r|d\theta|) \\ \leq & \frac{1}{|\lambda|^{5/6}} \left[\frac{6}{11\cdot 2^{11/6}} + \frac{21}{20\cdot 2^{5/6}} - \frac{3}{11\cdot 3^{11/6}} - \frac{21}{40\cdot 3^{5/6}} \right] \frac{1}{(1-1/2)^{5/2}} \\ + & \frac{1}{|\lambda|^{5/6}} \left[\frac{24\cdot 2^{2/3} + 21\cdot 2^{5/3}}{16\cdot 2^{15/6}} \right] \frac{\pi/3 + \phi/3}{(1-1/2)^{5/2}}. \end{aligned} \tag{47}$$

Since $|z(t)| \geq (2|\lambda|)^{1/3}$, by Lemma 19, we have that if $|z(t)| \leq (2|\lambda|)^{1/3}$, then $\pi/6 + \phi/3 \leq \arg z(t) \leq \pi/2$. When $\pi/6 + \phi/3 \leq \arg z(t) \leq \pi/2$, we know that $\pi + \phi \leq \arg(iz(t)^3) \leq 2\pi$. Thus $|\lambda - iz(t)^3| = |z|^3|1 - \lambda/(iz^3)| \geq r^3 \sin\phi$. Thus,

$$\begin{aligned} & \int_{|z(t)|\leq (2|\lambda|)^{1/3}} \left| \frac{3z(8\lambda + 7iz^3)}{16(\lambda - iz^3)^{5/2}} \right| (|dr| + r|d\theta|) \\ \leq & \frac{1}{|\lambda|^{5/6}\sin^{5/2}\phi} \left[2\int_{\sin(\pi/6+\phi/3)}^{2^{1/3}} \left(\frac{3}{2\rho^{13/2}} + \frac{21}{16\rho^{7/2}} \right) d\rho \right. \\ + & \left. \int_{\pi/6+\phi/3}^{\pi/2} \left(\frac{3}{2\rho^{11/2}} + \frac{21}{16\rho^{5/2}} \right) d\theta \right] \end{aligned} \tag{48}$$

$$\begin{aligned} \leq & \frac{1}{|\lambda|^{5/6}\sin^{5/2}\phi} \left[\frac{6}{11} \left(\frac{1}{2^{11/6}} - \frac{1}{\sin(\pi/6+\phi/3)^{11/2}} \right) \right. \\ + & \frac{21}{20} \left(\frac{1}{2^{5/6}} - \frac{1}{\sin(\pi/6+\phi/3)^{5/2}} \right) \\ + & \left. \left(\frac{3}{2\sin(\pi/6+\phi/3)^{11/2}} + \frac{21}{16\sin(\pi/6+\phi/3)^{5/2}} \right) (\pi/3 - \phi/3) \right], \end{aligned} \tag{49}$$

where $|\lambda|^{1/3}\sin(\pi/6+\phi/3)$ an obvious lower bound of $|z(t)|$ since $z(t)$ moves up to the left and since the inverse image of $|\operatorname{Im} Z|\le \operatorname{arcsinh}(1)$ intersects the ray $\{\arg z=\pi/6+\phi/3\}$ at $|z|\ge|\lambda|^{1/3}$. Also $\pi/2-(\pi/6+\phi/3)$ at the end of the last line reflects that $z(t)$ intersects the positive imaginary axis at $|z(t)|\ge(2|\lambda|)^{1/3}$. We let $B(\phi)$ be the sum of (47) and (49) that is an upper bound for $\int_Z^{Z+\infty}|F(Z+s)|ds$. ☐

5. Proof of Theorem 1

Proof. We will prove the contrapositive form of the statement for $\phi=\arg\lambda>0$. Suppose that (2) does not hold. First, note that

$$|\lambda|>R(\phi)=\left(\frac{B(\phi)}{\log(1+1/\sqrt{2})}\right)^{6/5}\quad\text{or}\quad \log(1+1/\sqrt{2})>\frac{1}{|\lambda|^{5/6}}B(\phi).$$

$$\begin{aligned}R(\phi)\ &\ge\ \left\{\frac{1}{\log(1+1/\sqrt{2})}\left(\frac{873\cdot 2^{2/3}-271\sqrt{2}\cdot 3^{1/6}}{330}+\frac{11(\pi+0)}{4\cdot 2^{1/3}}\right.\right.\\ &+\ \frac{1}{880\sin^{5/2}\phi}\left[582\cdot 2^{1/6}+\frac{-480+440\pi-440(\pi/5)}{\sin^{11/2}(\pi/6+(\pi/5)/3)}\right.\\ &+\ \left.\left.\left.\frac{-924+385\pi-385(\pi/5)}{\sin^{5/2}(\pi/6+(\pi/5)/3)}\right]\right)\right\}^{6/5}\\ &\ge\ (7.02+13.76/\sin^{5/2}\phi)^{6/5}\ge 133. \qquad (50)\end{aligned}$$

In Theorem 9, we choose $\delta_1=\pi/2$ and $\delta_2=\operatorname{arcsinh}(1)$. Then

$$1+\frac{\min\{\sin\delta_1,\sinh\delta_2\}}{\cosh\delta_2}=1+\frac{1}{\cosh(\operatorname{arcsinh}(1))}=1+\frac{1}{\sqrt{2}}.$$

Certainly, the existence of $\mathcal{G}$ is given by Lemma 14 with (50).

Since (2) does not hold, we have that $|\lambda|^{5/6}\sin\phi/\,2^{1/6}\ge\operatorname{arcsinh}(1)$. That is, by Lemma 13, the inverse image of $|\operatorname{Im} Z|\le\operatorname{arcsinh}(1)$ intersects the positive imaginary axis at $|z|\ge(2|\lambda|)^{1/3}$.

Also, if (2) does not hold, then the following hold

$$\operatorname{Im} Z((3|\lambda|)^{1/3}e^{(\pi/6+\phi/3)i})\le\operatorname{arcsinh}(1)\quad\text{and}\quad\operatorname{Im} Z(|\lambda|^{1/3}e^{(\pi/6+\phi/3)i})\ge\operatorname{arcsinh}(1), \qquad (51)$$

that imply the inverse image of $|\operatorname{Im} Z|\le\operatorname{arcsinh}(1)$ intersects the ray $\{\arg z=\pi/6+\phi/3\}$ at $|\lambda|^{1/3}\le|z|\le(3|\lambda|)^{1/3}$. In fact, one can show that $|\lambda|\ge 19$ is enough for this purpose. We will use this observation in the proof of Theorem 22.

Thus, by Theorem 21, if $Z\in\mathcal{G}$ with $|\operatorname{Im} Z|\le\operatorname{arcsinh}(1)$, then

$$\int_Z^{Z+\infty}|F(Z+s)|ds\le\frac{1}{|\lambda|^{5/6}}B(|\arg\lambda|)<\log(1+1/\sqrt{2}).$$

In particular, by Theorem 9, the transformed eigenfunction U has a zero in $O_1\subset\mathcal{G}$.

Now we claim that the inverse image $Z^{-1}(O_1)$ entirely lies in the zero-free region of the eigenfunction. This gives a contradiction because of $U(Z(z)) = (\lambda - iz^3)^{1/4}u(z)$, that is, if $Z(z_0)$ is a zero of U, then z_0 is a zero of u.

So it remains to show that if $|\lambda| \geq \left[\frac{B(\phi)}{\log(1+1/\sqrt{2})}\right]^{6/5} = R(\phi)$, then the inverse image $Z^{-1}(O_1)$ entirely lies in the zero-free region of the eigenfunction. Certainly, the left-hand side of O_1 is within $\mathcal{G}$. See Lemma 14.

We know that $\operatorname{Im} Z(\rho|\lambda|^{1/3}e^{(5\pi/12+\phi/3)i})$ is continuously decreasing for $\rho \geq 1$. Thus if we can show that $\operatorname{Im} Z(|\lambda|^{1/3}e^{(5\pi/12+\phi/3)i}) \geq \operatorname{arcsinh}(1)$ when (2) does not hold, then the inverse image $Z^{-1}(\{Z \in \mathcal{G} : |\operatorname{Im} Z| \leq \operatorname{arcsinh}(1)\})$ intersects $\{\arg z = 5\pi/12 + \phi/3\}$ only at $\rho \geq 1$.

From the remark after Lemma 15, we know that

$$\begin{aligned}
\operatorname{Im} Z(|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)}) &> |\lambda|^{5/6}L_1 \sin(5\phi/6 - 5\pi/24 + \arg K_1)\big|_{\rho=1} \\
&\geq 1.325|\lambda|^{5/6}\sin(5\phi/6 - 0.062) \\
&\geq \operatorname{arcsinh}(1) \quad \text{if} \quad \phi \geq \pi/20 \quad \text{and} \quad |\lambda| \geq 19.
\end{aligned}$$

Again from Remark after Lemma 15, we know that

$$\operatorname{Re} Z(|\lambda|^{1/3}e^{i(5\pi/12+\phi/3)}) \geq |\lambda|^{5/6}L_1 \cos(5\phi/6 - 5\pi/24 + \arg K_2)\big|_{\rho=1}.$$

Since $\max\{\operatorname{Re} Z : Z \in O_1\} = 7\pi/4$, if

$$|\lambda|^{5/6}L_1 \cos(5\phi/6 - 5\pi/24 + \arg K_2)\big|_{\rho=1} \geq 7\pi/4,$$

then the inverse image $Z^{-1}(O_1)$ entirely lies in the zero-free region of the eigenfunction for $\phi \geq \pi/20$.

If $|\lambda| \geq 19$ and $\pi/20 \leq \phi \leq \pi/5$, then using a calculator one can show that

$$|\lambda|^{5/6}L_1 \cos(5\phi/6 - 5\pi/24 + \arg K_2)\big|_{\rho=1} \geq 13 \geq 7\pi/4. \tag{52}$$

Hence if (2) does not hold and $\pi/20 \leq \phi \leq \pi/5$, then $Z^{-1}(O_1)$ entirely lie in the zero-free region of the eigenfunction which is a contradiction.

For $0 < \phi < \pi/20$, certainly the imaginary part of $z_0 = |\lambda|^{1/3}e^{i(5\pi/12+\phi/3)}$ is less than that of the lowest point $i(|\lambda|\cos\phi)^{1/3}$ of $\operatorname{Re}(\lambda - iz^3) = 0$.

Now we consider the inverse image under $z \mapsto Z$ of vertical line segments. Since $z \mapsto Z$ is analytic with a non-vanishing first derivative, the map preserves the angles. Hence from the fact that inverse images of horizontal lines move up to the left, we know that the inverse images of vertical lines move down to the left. Thus the inverse image ξ of the vertical line passing through z_0 does not meet with $\operatorname{Re}(\lambda - iz_0^3) < 0$ in $\{z \in \mathcal{D} : \operatorname{Re} z \leq \operatorname{Re} z_0\}$.

In Lemma 15, we see that $\rho^{5/2}L_1$ is increasing for $\rho \geq 1$. Thus, if we can show that for $0 < \phi < \pi/20$,

$$|\lambda|^{5/6}L_1 \cos\left(\max_{j=1,2} |5\phi/6 - 5\pi/24 + \arg K_j|\right)\Bigg|_{\rho=1} \geq 2 \cdot \frac{7\pi}{4},$$

then certainly $Z^{-1}(O_1)$ entirely lies in the zero-free region of the eigenfunction. This proves the claim.

Therefore, if λ is an eigenvalue of (1), $|\lambda| \le R(|\arg\lambda|)$. Then

$$\begin{aligned} R(\phi) &\le \left\{ \frac{1}{\log(1+1/\sqrt{2})} \left(\frac{873\cdot 2^{2/3} - 271\sqrt{2}\cdot 3^{1/6}}{330} + \frac{11(\pi+(\pi/5))}{4\cdot 2^{1/3}} \right.\right. \\ &+ \frac{1}{880\sin^{5/2}\phi}\left[582\cdot 2^{1/6} + \frac{-480+440\pi-440(0)}{\sin^{11/2}(\pi/6+(0)/3)} \right. \\ &+ \left.\left.\left. \frac{-924+385\pi-385(0)}{\sin^{5/2}(\pi/6+(0)/3)} \right]\right)\right\}^{6/5} \\ &\le (7.03+91.59/\sin^{5/2}\phi)^{6/5} \le 231.59/\sin^3\phi. \end{aligned}$$

So $|\mathrm{Im}\,\lambda|^3 \le 231.59|\lambda|^2 \le 231.59(\sec^2\frac{\pi}{5})|\mathrm{Re}\,\lambda|^2$. Hence, $|\mathrm{Im}\,\lambda| \le 7.1|\mathrm{Re}\,\lambda|^{2/3}$. ☐

6. Proof of Theorem 2

In the previous theorem, we used very rough estimates. Now with more accurate estimates, certainly we get better results on the eigenvalues. We begin with the following theorem:

Theorem 22. *Let $z(t)$ is a parameterization of the inverse image of a horizontal half line with $|\mathrm{Im}\,Z| \le \delta_2$. Suppose that $|z(t)|$ attains its minimum at some point in $\pi/6+\phi/3 \le \arg z(t) \le \pi/3+\phi/3$. And suppose that $|z(t)| \ge |\lambda|^{1/3}$ when $\arg z(t) = \pi/3+\phi/3$. Then*

$$\int_Z^{Z+\infty} |F(Z+s)|ds \le \frac{1}{|\lambda|^{5/6}} B_1(\rho_0,\rho_1,\rho_2,\phi), \tag{53}$$

where

$$\begin{aligned} B_1(\rho_0,\rho_1,\rho_2,\phi) &= \frac{1}{(1+1/3^2)^{5/4}}\left(\frac{3}{11}(\frac{1}{\rho_0^{11/2}} - \frac{1}{3^{11/6}}) + \frac{21}{40}(\frac{1}{\rho_0^{5/2}} - \frac{1}{3^{5/6}}) \right) \\ &+ \frac{1}{(1+1/\rho_1^6)^{5/4}}\left(\frac{3}{11}(\frac{1}{\rho_0^{11/2}} - \frac{1}{\rho_1^{11/2}}) + \frac{21}{40}(\frac{1}{\rho_0^{5/2}} - \frac{1}{\rho_1^{5/2}}) \right) \\ &+ \frac{3\cdot 2^{5/4}}{11}(\frac{1}{\rho_1^{11/2}} - \frac{1}{\rho_2^{11/2}}) + \frac{21\cdot 2^{1/4}}{20}(\frac{1}{\rho_1^{5/2}} - \frac{1}{\rho_2^{5/2}}) \\ &+ \frac{1}{(1-1/\rho_2^3)^{5/2}}(\frac{3}{11\rho_2^{11/2}} + \frac{21}{40\rho_2^{5/2}}) + \frac{3+21/8\rho_0^3}{2(1+1/3^2)^{5/4}\rho_0^{11/2}}\cdot\frac{\pi}{6} \\ &+ \frac{3+21/8\rho_1^3}{2\rho_1^{11/2}2^{-5/4}}\cdot\frac{\pi}{12} + \frac{3+21/8\rho_2^3}{2\rho_2^{11/2}(1-1/\rho_2^3)^{5/2}}(\frac{\pi}{12}+\frac{\phi}{3}), \end{aligned}$$

where $|z(t)| = r = |\lambda|^{1/3}\rho$, ρ_0 is the minimum ρ-value, ρ_1 is the ρ-value at $\arg z = \pi/3+\phi/3$, and ρ_2 is a smaller of 1.94 and the minimum ρ-value at $\arg z = 5\pi/12+\phi/3$.

Note that $B_1(\rho_0,\rho_1,\rho_2,\phi)$ is a decreasing function of $\rho_0 > 0$ and $\rho_1 > 0$.

Proof. Let $z(t) \in \mathcal{D}$ be a parameterization of the inverse image of a horizontal half line in $\mathcal{G}$ with $|\operatorname{Im} Z(z(t)| \leq \operatorname{arcsinh}(1)$ and $\operatorname{Re} Z(z(t)$ increasing.

Let $|z(t)| = r = |\lambda|^{1/3}\rho$ and ρ_0 the minimum of $|z(t)$ over all t and all such curves $z(\cdot)$. Also, let ρ_1 and ρ_2 be the minimums of $|z(t)|$ over all such curves $z(\cdot)$ when $\arg z(t)$ is $\pi/3+\phi/3$ and $5\pi/12+\phi/3$, respectively. For each curve $z(t)$ we let $\rho_0^*, \rho_1^*, \rho_2^*$ be the minimum of $|z(t)|$ over all t, $|z(t)|$ when $\arg z(t)$ is $\pi/3+\phi/3$ and $5\pi/12+\phi/3$, respectively.

Like we did in the proof of Theorem 1, we get that

$$\int_Z^{Z+\infty} |F(Z+s)|ds \leq \int_\Gamma \left| \frac{3z(8\lambda+7iz^3)}{16(\lambda-iz^3)^{5/2}} \right| (|dr| + r|d\theta|).$$

In this proof we will also use the following observations frequently without mentioning them.

$$\left| \frac{3z(8\lambda+7iz^3)}{16(\lambda-iz^3)^{5/2}} \right| \leq \begin{cases} \frac{3r(8|\lambda|+7r^3)}{16r^{15/2}(1+|\lambda|^2/r^6)^{5/4}} & \text{if} \quad \pi/6+\phi/3 \leq \arg z \leq \pi/3+\phi/3 \\ \frac{3r(8|\lambda|+7r^3)}{16\cdot 2^{-5/4}r^{15/2}} & \text{if} \quad \pi/3+\phi/3 \leq \arg z \leq 5\pi/12+\phi/3 \\ \frac{3r(8|\lambda|+7r^3)}{16r^{15/2}(1-|\lambda|/r^3)^{5/2}} & \text{if} \quad \arg z \geq 5\pi/12+\phi/3. \end{cases}$$

In all case, we use a triangle inequality for the numerators. When $\pi/6+\phi/3 \leq \arg z(t) \leq \pi/3+\phi/3$, the above reflects that the angle between λ and iz^3 is at least $\pi/2$. When $\pi/3+\phi/3 \leq \arg z(t) \leq 5\pi/12+\phi/3$, the angle between λ and iz^3 is at least $\pi/4$. And finally, the last one is by a triangle inequality.

We first estimate

$$\int_\Gamma \left| \frac{3z(\lambda+7/8iz^3)}{2(\lambda-iz^3)^{5/2}} \right| |dr|$$

$$\leq \int_{\{\pi/6+\phi/3 \leq \arg z \leq \pi/3+\phi/3\}} \left| \frac{3z(\lambda+7/8iz^3)}{2(\lambda-iz^3)^{5/2}} \right| |dr| \tag{54}$$

$$+ \int_{\{\pi/3+\phi/3 \leq \arg z \leq 5\pi/12+\phi/3\}} \left| \frac{3z(\lambda+7/8iz^3)}{2(\lambda-iz^3)^{5/2}} \right| |dr| \tag{55}$$

$$+ \int_{\{5\pi/12+\phi/3 \leq \arg z \leq \pi/2+\phi/3\}} \left| \frac{3z(\lambda+7/8iz^3)}{2(\lambda-iz^3)^{5/2}} \right| |dr|. \tag{56}$$

We can estimate (54) as follows: Note that if $|\lambda| \geq 19$, then $(3/2|\lambda|)^{1/3} \leq |z(t)| \leq (3|\lambda|)^{1/3}$ as we saw in the proof of Theorem 1.

$$\begin{aligned} &\int_{\{\pi/6+\phi/3 \leq \arg z \leq \pi/3+\phi/3\}} \left| \frac{3z(\lambda+7/8iz^3)}{2(\lambda-iz^3)^{5/2}} \right| |dr| \\ \leq\ & \int_{\{\pi/6+\phi/3 \leq \arg z \leq \pi/3+\phi/3\}} \frac{3|\lambda|r+21/8r^4}{2r^{15/2}|1+\lambda i/z^3|^{5/2}}|dr|, \quad \text{set} \quad r = |\lambda|^{1/3}\rho, \\ \leq\ & \frac{1}{|\lambda|^{5/6}} \left[\int_{\rho_0^*}^{3^{1/3}} \frac{3\rho+21/8\rho^4}{2\rho^{15/2}|1+1/3^2|^{5/4}} d\rho + \int_{\rho_0^*}^{\rho_1^*} \frac{3\rho+21/8\rho^4}{2\rho^{15/2}|1+1/(\rho_1^*)^6|^{5/4}} d\rho \right] \\ \leq\ & \frac{1}{|\lambda|^{5/6}} \left[\frac{1}{(1+1/3^2)^{5/4}} \int_{\rho_0}^{3^{1/3}} \frac{3\rho+21/8\rho^4}{2\rho^{15/2}} d\rho + \frac{1}{(1+1/\rho_1^6)^{5/4}} \int_{\rho_0}^{\rho_1} \frac{3\rho+21/8\rho^4}{2\rho^{15/2}} d\rho \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|\lambda|^{5/6}}\left[\frac{1}{(1+1/3^2)^{5/4}}\left(\frac{3}{11}\left(\frac{1}{(\rho_0^*)^{11/2}}-\frac{1}{3^{11/6}}\right)+\frac{21}{40}\left(\frac{1}{(\rho_0^*)^{5/2}}-\frac{1}{3^{5/6}}\right)\right)\right.\\
&+ \left.\frac{1}{(1+1/(\rho_1^*)^6)^{5/4}}\left(\frac{3}{11}\left(\frac{1}{(\rho_0^*)^{11/2}}-\frac{1}{(\rho_1^*)^{11/2}}\right)+\frac{21}{40}\left(\frac{1}{(\rho_0^*)^{5/2}}-\frac{1}{(\rho_1^*)^{5/2}}\right)\right)\right]
\end{aligned}$$

where we use the substitution $r=|\lambda|^{1/3}\rho$ and ρ_0^* is the minimum ρ-value and ρ_1^* is the ρ-value at $\{\arg z=\pi/3+\phi/3\}$.

Similarly, the sum of (55) and (56) is bounded above by

$$\begin{aligned}
&\frac{1}{|\lambda|^{5/6}}\left[\int_{\rho_1^*}^{\rho_2^*}\left|\frac{3\rho+21/8\rho^4}{2\rho^{15/2}2^{-5/4}}\right|d\rho+\int_{\rho_2^*}^{\infty}\left|\frac{3\rho+21/8\rho^4}{2\rho^{15/2}(1-1/(\rho_2^*)^3)^{5/2}}\right|d\rho\right]\\
&= \frac{1}{|\lambda|^{5/6}}\left[\frac{3\cdot 2^{5/4}}{11}\left(\frac{1}{(\rho_1^*)^{11/2}}-\frac{1}{(\rho_2^*)^{11/2}}\right)+\frac{21\cdot 2^{1/4}}{20}\left(\frac{1}{(\rho_1^*)^{5/2}}-\frac{1}{(\rho_2^*)^{5/2}}\right)\right.\\
&+ \left.\frac{1}{(1-1/(\rho_2^*)^3)^{5/2}}\left(\frac{3}{11(\rho_2^*)^{11/2}}+\frac{21}{40(\rho_2^*)^{5/2}}\right)\right],
\end{aligned}$$

where ρ_2^* is the ρ-value at $\{\arg z=5\pi/12+\phi/3\}$.

Similarly, we can estimate

$$\int_\Gamma\left|\frac{3z(\lambda+7/8iz^3)}{2(\lambda-iz^3)^{5/2}}\right|r|d\theta|$$

as follows:

$$\begin{aligned}
&\int_\Gamma\left|\frac{3z(\lambda+7/8iz^3)}{2(\lambda-iz^3)^{5/2}}\right|r|d\theta|\leq\int_{\pi/6+\phi/3}^{\pi/3+\phi/3}\frac{3r^2|\lambda|+21/8r^5}{2r^{15/2}(1+|\lambda|^2/r^6)^{5/4}}d\theta\\
&+ \int_{\pi/3+\phi/3}^{5\pi/12+\phi/3}\frac{3r^2|\lambda|+21/8r^5}{2r^{15/2}2^{-5/4}}d\theta+\int_{\{5\pi/12+\phi/3\leq\theta\leq\pi/2+\phi/3\}}\frac{3r^2|\lambda|+21/8r^5}{2r^{15/2}(1-|\lambda|/r^3)^{5/2}}|d\theta|\\
&\leq \frac{1}{|\lambda|^{5/6}}\left[\frac{3+21/8(\rho_0^*)^3}{2(\rho_0^*)^{11/2}(1+1/3^2)^{5/4}}\frac{\pi}{6}+\frac{3+21/8(\rho_1^*)^3}{2(\rho_1^*)^{11/2}2^{-5/4}}\frac{\pi}{12}\right.\\
&+ \left.\frac{3+21/8(\rho_2^*)^3}{2(\rho_2^*)^{11/2}(1-1/(\rho_2^*)^3)^{5/2}}(\pi/12+\phi/3).\right]
\end{aligned}$$

Thus we have

$$\int_Z^{Z+\infty}|F(Z+s)|ds\leq\frac{1}{|\lambda|^{5/6}}B_1(\rho_0^*,\rho_1^*,\rho_2^*,\phi),\quad\text{for each}\quad|\operatorname{Im}Z|\leq\operatorname{arcsinh}(1).$$

Since $\rho_0^*\geq\rho_0$ and $\rho_1^*\geq\rho_1$, it is easy to see that

$$B_1(\rho_0^*,\rho_1^*,\rho_2^*,\phi)\leq B_1(\rho_0,\rho_1,\rho_2^*,\phi).$$

That is, B_1 is decreasing for positive ρ_0^* and ρ_1^*.

Fix ρ_0,ρ_1 and ϕ. Then

$$\begin{aligned}
B_1(\rho_0,\rho_1,\rho_2^*,\phi)&=const.+\frac{21}{(\rho_2^*)^{5/2}}\left(\frac{1/40+1/16(\pi/12+\phi/3)}{(1-1/(\rho_2^*)^3)^{5/2}}-\frac{2^{1/4}}{20}\right)\\
&+ \frac{3}{(\rho_2^*)^{11/2}}\left(\frac{1}{(1-1/(\rho_2^*)^3)^{5/2}}\left(\frac{1}{11}+\frac{\pi}{24}+\frac{\phi}{6}\right)-\frac{2^{5/4}}{11}\right).
\end{aligned}$$

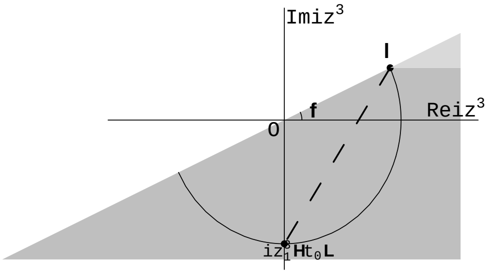

FIGURE 6. The radius of the circle is $|\lambda|$ on which λ and $iz_1^3(t_0)$ lie.

One can see that the coefficient of $3/(\rho_2^*)^{11/2}$ is decreasing and positive by a straight forward computation for all $\rho_2^* > 1$. Also, we see here that the coefficient of $21/(\rho_2^*)^{5/2}$ is decreasing for all $\rho_2^* > 1$ and remains positive if $1 < \rho_2^* < 1.94$. Thus the above is decreasing for $1 < \rho_2^* < 1.94$. Also if $\rho_2^* \geq 1.94$, then

$$B_1(\rho_0, \rho_1, \rho_2^*, \phi) \leq B_1(\rho_0, \rho_1, 1.94, \phi) \leq B_1(\rho_0, \rho_1, \rho_2, \phi) \quad \text{if} \quad \rho_2 \leq 1.94.$$

Therefore, (53) holds since ρ_2 is a number less than 1.94 and the minimum ρ-value at $\arg z = 5\pi/12 + \phi/3$. ☐

Note that we do not need to compute exact ρ_j's to find an upper bound of the integral in (53), but we need to find only lower bounds of ρ_0, ρ_1 and ρ_2. In the following lemma, we find a lower bound of ρ_0. Certainly, the better lower bound of ρ_0 gives the better estimate of the integral in (53).

Lemma 23. *If* $|\lambda| \geq 19$, *then* $\rho_0 \geq \sin(23\pi/36 - 5\phi/6)$ *for* $\pi/6 \leq \phi \leq \pi/5$, *and for each inverse image* $z(t)$, *the minimum of* $|z(t)|$ *over* t *occurs at* $\arg z < 5\pi/18 + \phi/3$.

If $|\lambda| \geq 145$, *then* $\rho_0 \geq \sin(13\pi/24 - 5\phi/6)$ *for* $\pi/20 \leq \phi < \pi/6$, *and for each inverse image* $z(t)$, *the minimum of* $|z(t)|$ *over* t *occurs at* $\arg z < \pi/3 + \phi/3$.

Proof. Suppose that $\pi/6 \leq \phi \leq \pi/5$ and $|\lambda| \geq 19$. Then by Lemma 18, we have

$$\operatorname{Im} Z(|\lambda|^{1/3} e^{i(5\pi/18+\phi/3)}) \geq 0.842 \cdot 19^{5/6} \sin(5\phi/36 - 0.047) > 3.717 > \operatorname{arcsinh}(1).$$

Let $z_1(t)$ be the inverse image of a horizontal line with $z_1(t_0) = |\lambda|^{1/3} e^{(5\pi/18+\phi/3)i}$ for some t_0. Since $\operatorname{Im} Z(z_1(t)) > \operatorname{arcsinh}(1)$, we see that a part of $Z(z_1(t))$ does not lie in $\mathcal{G}$, but since $Z(z_1(t_0))$ lies in $Z(\mathcal{D})$, $z_1(t)$ exists until it hits the boundary of $\mathcal{G}$. In particular, $z_1(t)$ exists for $\pi/6 + \phi/3 \leq \arg z_1(t) \leq \pi/2$.

Let $z(t)$ be the inverse image of a horizontal line with $|\operatorname{Im} Z(z(t)) \leq \operatorname{arcsinh}(1)$. Then it is clear that

$$\min_t |z(t)| \geq \min_t |z_1(t)|. \tag{57}$$

From now on we will estimate $\min_t |z_1(t)|$ which gives a lower bound of $\min_t |z(t)|$. We see that the angle between λ and $iz_1^3(t_0)$ is $2\pi/3$. These two points with the origin form an isosceles triangle. Thus $\arg(\lambda - iz_1^3(t_0)) = \pi/6 + \phi$. Apply this to (38). Then

$$\arg \frac{\overline{z_1(t_0)}}{\sqrt{\lambda - iz_1(t_0)^3}} = -\arg z_1(t_0) - (\pi + 1/2 \arg(\lambda - iz_1(t_0)^3)) \leq -\frac{3\pi}{2},$$

where the equality holds for $\phi = \pi/6$. So if $\phi = \pi/6$, $\min_t |z_1(t)| = |\lambda|^{1/3}$. Also, by Lemma 19, $|z_1(t)| \geq |\lambda|^{1/3}$ if $t \geq 1$. If $\phi > \pi/6$ then $|z_1(t)| < |\lambda|^{1/3}$ for some $t < t_0$. While $|z_1(t)| < |\lambda|^{1/3}$, since $\arg z_1(t) < 5\pi/18 + \phi/3$, we see that $\arg(\lambda - iz_1^3(t_0)) < \pi/6 + \phi$. Then from Lemma 4,

$$\arg z_1'(t) \geq \arg z_1'(t_0) = \pi - \frac{\arg(\lambda - iz_1^3(t_0))}{2} = \frac{11\pi}{12} - \frac{\phi}{2}.$$

Thus the distance from the origin to the line passing through $z_1(t_0)$ with the slope $\tan(11\pi/12 - \phi/2)$ is smaller than $\min_t |z_1(t)|$. Then using the elementary fact that the distance from the origin to the line $ax + by + c = 0$ for some real a, b, c is $|c|/\sqrt{a^2 + b^2}$, one can get the desired lower bound for $\min_t |z_1(t)|$.

Since $|z_1(t)|$ is increasing at t_0, one can see that for each inverse image $z(t)$ of a horizontal line in $|\operatorname{Im} Z| \leq \operatorname{arcsinh}(1)$, $|z(t)|$ is increasing at $\arg z = 5\pi/18 + \phi/3$. So the minimum occurs at $\arg z \leq 5\pi/18 + \phi/3$.

Similarly, if $|\lambda| \geq 145$, then using $\arg z = \pi/3 + \phi/3$ instead of $\arg z = 5\pi/18 + \phi/3$, one can show the second part of this lemma. We omit the details. ☐

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Suppose that $|\lambda| \geq 19$ for $\pi/6 \leq \arg \lambda \leq \pi/5$. Suppose also that $|\lambda| \geq 145$ for $\pi/10 \leq \arg \lambda < \pi/6$. Like we did in the proof of Theorem 1, we choose $\delta_1 = \pi/2$ and $\delta_2 = \operatorname{arcsinh}(1)$ in Theorem 9. Then

$$1 + \frac{\min\{\sin \delta_1, \sinh \delta_2\}}{\cosh \delta_2} = 1 + \frac{1}{\sqrt{2}}.$$

We first look at the image of $\arg z = 5\pi/12 + \phi/3$. If $|\lambda| \geq 19$, then the inverse image of $\{Z \in \mathcal{G} : |\operatorname{Im} Z| \leq \operatorname{arcsinh}(1)\}$ intersects $\arg z = 5\pi/12 + \phi/3$ at

$$\rho \geq \rho_2 \geq 1.7246 \quad \text{for all} \quad \pi/6 \leq \phi \leq \pi/5 \tag{58}$$

because, by Lemma 15,

$$\begin{aligned} \operatorname{Im} Z(1.724619^{1/3} e^{(5\pi/12 + \phi/3)i}) &\geq 25.239 \sin(5\phi/6 - 0.402) \\ &\geq 0.887 \geq \operatorname{arcsinh}(1) \approx 0.8813736 \quad \text{for} \quad \pi/6 \leq \phi \leq \pi/5. \end{aligned}$$

If $|\lambda| \geq 145$, then the inverse image of $\{Z \in \mathcal{G} : |\operatorname{Im} Z| \leq \operatorname{arcsinh}(1)\}$ intersects $\arg z = 5\pi/12 + \phi/3$ at

$$\rho \geq \rho_2 \geq 1.3546 \quad \text{for all} \quad \pi/10 \leq \phi < \pi/6 \tag{59}$$

because, by Lemma 15,

$$\begin{aligned} \operatorname{Im} Z(1.3546145^{1/3} e^{(5\pi/12+\phi/3)i}) &\geq 100.403 \sin(5\phi/6 - 0.253) \\ &\geq 0.889 \geq \operatorname{arcsinh}(1) \quad \text{for} \quad \pi/10 \leq \phi < \pi/6. \end{aligned}$$

Now we estimate the image of $\arg z = \pi/3 + \phi/3$. If $|\lambda| \geq 19$, then the inverse image of $\{Z \in \mathcal{G} : |\operatorname{Im} Z| \leq \operatorname{arcsinh}(1)\}$ intersects $\arg z = \pi/3 + \phi/3$ at

$$\rho \geq \rho_2 \geq 1.2298 \quad \text{for all} \quad \pi/6 \leq \phi \leq \pi/5 \tag{60}$$

because, by Lemma 17,

$$\begin{aligned} \operatorname{Im} Z(1.229819^{1/3} e^{(5\pi/12+\phi/3)i}) &\geq 12.934 \sin(5\phi/6 - 0.368) \\ &\geq 0.895 \geq \operatorname{arcsinh}(1) \approx 0.8813736 \quad \text{for} \quad \pi/6 \leq \phi \leq \pi/5. \end{aligned}$$

If $|\lambda| \geq 145$, then the inverse image of $\{Z \in \mathcal{G} : |\operatorname{Im} Z| \leq \operatorname{arcsinh}(1)\}$ intersects $\arg z = \pi/3 + \phi/3$ at

$$\rho \geq \rho_2 \geq 1.1323 \quad \text{for all} \quad \pi/10 \leq \phi < \pi/6 \tag{61}$$

because, by Lemma 17,

$$\begin{aligned} \operatorname{Im} Z(1.1323145^{1/3} e^{(5\pi/12+\phi/3)i}) &\geq 67.843 \sin(5\phi/6 - 0.249) \\ &\geq 0.895 \geq \operatorname{arcsinh}(1) \quad \text{for} \quad \pi/10 \leq \phi < \pi/6. \end{aligned}$$

Also by Lemma 23, we have that if $|\lambda| \geq 14$, then

$$\begin{aligned} \rho_0 &\geq \sin(23\pi/36 - 5\phi/6) \quad \text{if} \quad \pi/6 \leq \phi \leq \pi/5, \quad \text{and} \\ \rho_0 &\geq \sin(13\pi/24 - 5\phi/6) \quad \text{if} \quad \pi/10 \leq \phi < \pi/6. \end{aligned}$$

Finally, we look at the image of $\arg z = \pi/6 + \phi/3$. From Lemma 11, we see that

$$\operatorname{Im} Z((3/2|\lambda|)^{1/3} e^{i(\pi/6+\phi/3)}) \geq \frac{79|\lambda|^{5/6}}{105 \cdot 2^{5/6} 3^{1/6}}$$

and

$$\operatorname{Im} Z((3|\lambda|)^{1/3} e^{i(\pi/6+\phi/3)}) \leq \frac{|\lambda|^{5/6}}{5 \cdot 3^{1/6}}.$$

From the second inequality we have condition that

$$|\lambda| \geq \left(5 \cdot 3^{1/6} \operatorname{arcsinh}(1)\right)^{6/5} \geq 7.39, \tag{62}$$

in order to have $\operatorname{Im} Z((3|\lambda|)^{1/3} e^{i(\pi/6+\phi/3)}) \leq \operatorname{arcsinh}(1)$. And (62) is enough to insure that $\operatorname{Im} Z((3/2|\lambda|)^{1/3} e^{i(\pi/6+\phi/3)}) \geq \operatorname{arcsinh}(1)$.

Let $Z \in \mathcal{G}$ with $|\mathrm{Im}\, Z| \leq \mathrm{arcsinh}(1)$. If $|\lambda| \geq 19$ and $\pi/6 \leq \phi \leq \pi/5$, then we have (53). Moreover, we claim that

$$B_1(\rho_0, \rho_1, \rho_2, \phi) \leq B_1(\sin(23\pi/36 - 5(\pi/5)/6), 1.2298, 1.7246, \pi/5) \leq 6.2181543. \tag{63}$$

Also, if $|\lambda| \geq 145$ and $\pi/10 \leq \phi < \pi/6$, then we have (53). Moreover, we claim that

$$B_1(\rho_0, \rho_1, \rho_2, \phi) \leq B_1(\sin(13\pi/24 - 5(\pi/6)/6), 1.1323, 1.3546, \pi/6) \leq 32.3870100. \tag{64}$$

Certainly as it is explained in the proof of Theorem 22, $B_1(\rho_0, \rho_1, \rho_2, \phi)$ is decreasing for $\rho_0 > 0, \rho_1 > 0$ and $\rho_2 > 1$. Hence, we have that if $\pi/6 \leq \phi \leq \pi/5$, then

$$\int_Z^{Z+\infty} |F(Z+s)|ds \leq \frac{1}{|\lambda|^{5/6}} B_1(\sin(23\pi/36 - 5\phi/6), 1.2298, 1.7246, \phi),$$

where the right hand side is maximized at $\phi = \pi/5$.

Also, if $\pi/10 \leq \phi < \pi/6$, then

$$\int_Z^{Z+\infty} |F(Z+s)|ds \leq \frac{1}{|\lambda|^{5/6}} B_1(\sin(13\pi/24 - 5\phi/6), 1.1323, 1.3546, \phi),$$

where the right hand side is maximized at $\phi = \pi/6$. This proves the claim.

Thus if $|\lambda| \geq 19$ for $\pi/6 \leq \phi \leq \pi/5$, then

$$\int_Z^{Z+\infty} |F(Z+s)|ds \leq \frac{6.2181543}{19^{5/6}} \leq 0.5346056 < 0.5347999 < \log(1 + \frac{1}{\sqrt{2}}).$$

Hence, the transformed solution $U(Z)$ has exactly one zero in O_1.

Also, the inverse image $Z^{-1}(O_1)$ entirely lies in the zero-free region of the eigenfunction u by (52). This is a contradiction. Therefore, if $\pi/6 \leq \arg \lambda \leq \pi/5$, then $|\lambda| < 19$. Since $\overline{\lambda}$ is also an eigenvalue if λ is, we conclude that

$$\text{if} \quad |\lambda| \geq 19, \quad \text{then} \quad |\arg \lambda| < \frac{\pi}{6}.$$

Similarly, if $|\lambda| \geq 145$ for $\pi/10 \leq \phi < \pi/6$, then since

$$\int_Z^{Z+\infty} |F(Z+s)|ds \leq \frac{32.3870100}{145^{5/6}} \leq 0.5119539 < 0.5347999 < \log(1 + \frac{1}{\sqrt{2}}),$$

we conclude that

$$\text{if} \quad |\lambda| \geq 145, \quad \text{then} \quad |\arg \lambda| < \frac{\pi}{10}.$$

□

Remark. Theorem 2 improves Theorem 1 but the author does not claim that Theorem 2 is the best result by the method we use in this paper. There are various places in the proofs one can slightly improve Theorem 2.

The next theorem is due to Mezincescu. Since it is not published, the author includes Mezincescu's proof.

Theorem 24. *Let u be an eigenfunction of (1) with the eigenvalue λ. Then* $\mathrm{Re}\, \lambda \geq \max_{0<\theta<\pi/10} \left[|\mathrm{Im}\, \lambda| \sin 3\theta + 5(\frac{\cos 5\theta}{12})^{3/5} \right] / \cos 3\theta$. *In particular,* $\mathrm{Re}\, \lambda \geq 5/12^{3/5} \approx 1.1258$.

Proof. We consider the parameterized line $z(t) = y_0 + (t - y_0 \cos\theta)e^{i\theta}$, $-\infty < t < \infty$, where y_0 is to be chosen later. Let $f(t) = u(z(t))$. Then from (1), we see that $f(t)$ satisfies that

$$-f''(t) + e^{2i\theta}(iz(t)^3 - \lambda)f(t) = 0.$$

Also if $-\pi/10 < \theta < \pi/10$, both ends of $z(t)$ stay in decaying Stokes sectors. Thus we have integrability that follows. We now multiply the above by $\overline{f(t)}$ and integrate over $-\infty < t < \infty$. Then

$$\int_{-\infty}^{\infty} |f'(t)|^2 dt \tag{65}$$

$\square$

7. Conclusions

Applying to our problem Hille's asymptotic integration method and the Liouville transformation for a bounded region, we proved that if λ is an eigenvalue of (1), then $|\mathrm{Im}\,\lambda| \le (7.1)(\mathrm{Re}\,\lambda)^{2/3}$. So in particular, $\arg\lambda \to 0$ as $|\lambda| \to \infty$. Then we improve this by reducing the range of $\arg\lambda$. That is, if $\pi/6 \le |\arg\lambda| \le \pi/5$, then $|\lambda| < 19$, and if $\pi/10 \le |\arg\lambda| < \pi/5$, then $|\lambda| < 145$.

The method we used here can be applied to other eigenproblems with simple potential terms. If the potential term is complicated, it might be difficult to trace a zero-free region of an eigenfunction and the inverse image of a horizontal line under the Liouville transformation. Also, this method gives a way of locating zeros in a bounded region for other problems.

Obvious open problems are to reduce the region where the eigenvalues should lie, and finally to prove that the eigenvalues are real. Then one might want to show the existence of an eigenfunction. If the reality and existence were settled, this would give a new example of a second order with a polynomial potential that has a solution with infinitely many real zeros and only finitely many non-real zeros (see [].)

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